Small-Signal Stability Analysis of Neutral Delay Differential Equations

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***Abstract*— This paper focuses on the small-signal stability analysis of systems modeled as Neutral Delay Differential Equations (NDDEs). These systems include delays in both the state variables and their first time derivatives. The proposed approach consists in descriptor model transformation that constructs an equivalent set of Delay Differential Algebraic Equations (DDAEs) of the original NDDE. The resulting DDAE is a non-index-1 Hessenberg form, whose characteristic equation consists of a series of infinite terms corresponding to infinitely many delays. Then, the effect on small-signal stability analysis is evaluated numerically through a Chebyshev discretization of the characteristic equations. Numerical appraisals focus on a variety of physical systems, including a population-growth model, a partial element equivalent circuit and a neutral delayed neural network.**

***Index Terms*— Time delay, delay differential algebraic equations (DDAEs), neutral time-delay differential equations (NDDEs), small-signal stability, Chebyshev discretization.**

I. Introduction

This paper focuses on the evaluation of the small-signal stability of Neutral Time-delay Differential Equations (NDDEs) in the form:

$$\mathbf{0} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{x}(t-\tau), \dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}(t-\tau)) \,, \tag{1}$$

i.e., differential equations where the delays appear in both the state variables and in their time derivatives. Systems in the form of NDDEs have wide applications in applied mathematics [1], [2], physics [3], ecology [4], [5], engineering [6], [7] and neural networks [8]–[10]. Conventional approaches for the stability analysis of a NDDE are based on Lyapunov-Krasovskii Functional (LKF) techniques [8]–[13]. This technique requires the solution of a Linear Matrix Inequality (LMI) problem, which is computationally demanding but has been recently made more tractable thanks to the LMI-Matlab Control Toolbox. However, due to the complexity and conservativeness of LKF, we believe a general and efficient methodology to study the stability of NDDEs is still missing. This paper presents a systematic approach to evaluate the small signal stability of NDDEs.

Reference [11] provides a descriptor model transformation approach that shows the stability of NDDEs (1) is consistent with the comparison set of non index-1 Hessenberg form Delay Differential Algebraic Equations (DDAEs), as shown in Section II. With the descriptor model transformation approach, we can develop the general stability analysis method of DDAEs to study the NDDEs.

Apart from LKFs-based approaches, some frequency domain approaches are also developed [16]–[20]. Reference [16] provides systematic eigenvalue-based methods for DDAEs. This approach is then further developed to solve the small-signal stability of power systems [21], [22]. Basing on [16], reference [21] improves the computation efficiency and simplifies the implement of the eigenvalue-based approach through using Chebyshev discretization to obtain the dominating eigenvalues. The Chebyshev discretization method is proved to achieve the best ratio of accuracy/computational burdens when studying the stability of large-size systems, i.e., real-world power system, in [23]. Especially, the Chebyshev discretization has also been applied to non index-1 Hessenberg form DDAEs [24]. This paper is based on the results of [24] as the equivalent DDAE in which the NDDE is transformed is a non-index-1 Hessenberg form.

Comparing with LKFs-based approach, the numerical approach provided in this paper has following advantages:

- *Accuracy.* The LKF approach provides only a sufficient stability condition. Therefore, the stability assertions obtained with the LKF-based approach tends to be conservative. The eigenvalue-based approach, on the other hand, provides sufficient and necessary stability conditions, and it is thus expected to predict more accurately the stability margin of the system than the LKF approach.
- *Efficiency.* The LKF approach requires the solution of a Linear Matrix Inequality (LMI) problem. The computational burden of LMI problems highly increase with the size of the system. The numerical complexity of the solution of the eigenvalue problem of large sparse matrices, however, does not increase as much as that of LMI problems, at least if only a reduced number of critical eigenvalues is computed.
- *Generality.* For complex non-linear systems, defining the Lyapunov function can be a challenge. Clearly, nonlinearity is not an issue for the small-signal stability analysis, which is based on the linearisation at an equilibrium point. Linearisation can be a limitation in those cases that require tracking the global stability but is nevertheless a stability "workhorse" in several engineering applications. The examples inlcuded in this paper also shows that using a proper parametric analysis, the small-signal stability analysis can lead to more accurate results than the LKF approach.

The remainder of the paper is organized as follows. Section

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II derives the expression of the characteristic equation of NDDEs based on the transformation into a non-index-1 Hessenberg form DDAE. Section III presents several examples and the corresponding numerical appraisal based on physical NDDE systems proposed in the literature. Conclusions are drawn in Section IV.

II. Small-signal Stability of NDDEs

This section defines the characteristic equation of (1), considering the single-delay case. The extension to the multiple-delay case is straightforward. To simplify the development of the proofs included in this section, let

$$\boldsymbol{x}_d = \boldsymbol{x}(t-\tau) \tag{2}$$

be the *retarded* or *delayed* state and algebraic variables, respectively, where t is the current simulation time, and τ ($\tau > 0$) is the time delay. In the remainder of this paper, since the main focus is on small-signal stability analysis, time delays are assumed to be constant.

Based on (2), (1) is rewritten as:

$$\mathbf{0}_{p,1} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{x}_d, \dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}_d) \,, \tag{3}$$

where $\boldsymbol{f}$ ($\boldsymbol{f} : \mathbb{R}^p \mapsto \mathbb{R}^p$) are the differential equations and $\boldsymbol{x}$ ($\boldsymbol{x} \in \mathbb{R}^p$) are the state variables. We also assume that (3) is autonomous, i.e., does not depend explicitly on time t. $\mathbf{0}_{i,j}$ denotes the zero matrix of i rows and j columns.

Since we are interested in the small-signal stability analysis, we consider only steady-state conditions and we linearize (3) at the equilibrium point:

$$\mathbf{0}_{p,1} = \boldsymbol{f}_{\boldsymbol{x}} \Delta\boldsymbol{x} + \boldsymbol{f}_{\boldsymbol{x}_d} \Delta\boldsymbol{x}_d + \boldsymbol{f}_{\dot{\boldsymbol{x}}} \Delta\dot{\boldsymbol{x}} + \boldsymbol{f}_{\dot{\boldsymbol{x}}_d} \Delta\dot{\boldsymbol{x}}_d \tag{4}$$

The *characteristic equation* of (4) is given by

$$\det \Delta(\lambda) = 0 \tag{5}$$

where

$$\Delta(\lambda) = \lambda(\boldsymbol{f}_{\dot{\boldsymbol{x}}} + e^{-\lambda\tau} \boldsymbol{f}_{\dot{\boldsymbol{x}}_d}) + \boldsymbol{f}_{\boldsymbol{x}} + e^{-\lambda\tau} \boldsymbol{f}_{\boldsymbol{x}_d} \tag{6}$$

is the *characteristic matrix* [**?**]. The solutions of (5) are called the *characteristic roots* or *spectrum*.

Instead of solving (6) directly, we propose to solve an equivalent characteristic equation, which is determined based on a variable transformation of the original NDDE (3). Let $\boldsymbol{y} = \dot{\boldsymbol{x}}$ and $\boldsymbol{f}_{\dot{\boldsymbol{x}}}$ be full rank, then (3) can be rewritten as:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{y} \\ \mathbf{0}_{p,1} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{x}_d, \boldsymbol{y}, \boldsymbol{y}_d) \,, \end{aligned} \tag{7}$$

which is a set of DDAEs. This is a typical descriptor model transformation [11].

Differentiating (3) at the equilibrium point leads to:

$$\begin{aligned} \Delta\dot{\boldsymbol{x}} &= \Delta\boldsymbol{y} \\ \mathbf{0}_{p,1} &= \boldsymbol{f}_{\boldsymbol{x}} \Delta\boldsymbol{x} + \boldsymbol{f}_{\boldsymbol{x}_d} \Delta\boldsymbol{x}_d + \boldsymbol{f}_{\boldsymbol{y}} \Delta\boldsymbol{y} + \boldsymbol{f}_{\boldsymbol{y}_d} \Delta\boldsymbol{y}_d \,, \end{aligned} \tag{8}$$

respectively, where $\boldsymbol{f}_{\boldsymbol{y}} \equiv \boldsymbol{f}_{\dot{\boldsymbol{x}}}$ and $\boldsymbol{f}_{\boldsymbol{y}_d} \equiv \boldsymbol{f}_{\dot{\boldsymbol{x}}_d}$. Note that, if $\boldsymbol{f}_{\boldsymbol{y}_d} \neq \mathbf{0}_{p,p}$, then (7) is a set of non-index-1 Hessenberg form DDAEs.

The derivation of the characteristic equation of general non-index-1 Hessenberg form DDAEs is thoroughly discussed in [24]. Such DDAEs have the following characteristic matrix:

$$\Delta(\lambda) = \lambda \boldsymbol{I}_p - \boldsymbol{A}_0 - e^{-\lambda\tau} \boldsymbol{A}_1 - \sum_{k=2}^{\infty} e^{-\lambda k\tau} \boldsymbol{A}_k \tag{9}$$

where $\boldsymbol{I}_p$ is the identity matrix of order p, and based on the results of [24] and the specific form of (8), one has:

$$\boldsymbol{A}_0 = \boldsymbol{A} \,, \tag{10}$$

$$\boldsymbol{A}_1 = \boldsymbol{D} \,, \tag{11}$$

$$\boldsymbol{A}_k = \boldsymbol{C}^{k-1} \boldsymbol{D}, \quad k \geq 2 \tag{12}$$

and

$$\begin{aligned} \boldsymbol{A} &= -\boldsymbol{f}_{\boldsymbol{y}}^{-1} \boldsymbol{f}_{\boldsymbol{x}} \,, & \boldsymbol{B} &= -\boldsymbol{f}_{\boldsymbol{y}}^{-1} \boldsymbol{f}_{\boldsymbol{x}_d} \,, \\ \boldsymbol{C} &= -\boldsymbol{f}_{\boldsymbol{y}}^{-1} \boldsymbol{f}_{\boldsymbol{y}_d} \,, & \boldsymbol{D} &= \boldsymbol{B} + \boldsymbol{C}\boldsymbol{A} \end{aligned} \tag{13}$$

Where $\boldsymbol{f}_{\boldsymbol{y}}^{-1}$ certainly exists as $\boldsymbol{f}_{\boldsymbol{y}} = \boldsymbol{f}_{\dot{\boldsymbol{x}}}$ is assumed to be full rank. The assumption that $\boldsymbol{f}_{\boldsymbol{y}}$ is full rank does not reduce the generality of the approach proposed in this paper. In fact, if $\boldsymbol{f}_{\dot{\boldsymbol{x}}}$ has rank q, $q < p$, (4) can be always rewritten as a set of DDAEs for which the Jacobian matrix $\tilde{\boldsymbol{f}}_{\dot{\tilde{\boldsymbol{x}}}}$ with respect of a subset of the state variables $\tilde{\boldsymbol{x}} \in \boldsymbol{x}$ is full rank. Moreover, if $\boldsymbol{f}_{\boldsymbol{y}}$ is full rank, (4) can be rewritten in an explicit form by multiplying by $-\boldsymbol{f}_{\boldsymbol{y}}^{-1}$:

$$\begin{aligned} \Delta\dot{\boldsymbol{x}} &= \Delta\boldsymbol{y} \\ \mathbf{0}_{p,1} &= \hat{\boldsymbol{f}}_{\boldsymbol{x}} \Delta\boldsymbol{x} + \hat{\boldsymbol{f}}_{\boldsymbol{x}_d} \Delta\boldsymbol{x}_d - \Delta\boldsymbol{y} + \hat{\boldsymbol{f}}_{\boldsymbol{y}_d} \Delta\boldsymbol{y}_d \,, \end{aligned} \tag{14}$$

Hence, (13) become:

$$\begin{aligned} \boldsymbol{A} &= \hat{\boldsymbol{f}}_{\boldsymbol{x}} \,, & \boldsymbol{B} &= \hat{\boldsymbol{f}}_{\boldsymbol{x}_d} \,, \\ \boldsymbol{C} &= \hat{\boldsymbol{f}}_{\boldsymbol{y}_d} \,, & \boldsymbol{D} &= \hat{\boldsymbol{f}}_{\boldsymbol{x}_d} + \hat{\boldsymbol{f}}_{\boldsymbol{y}_d} \hat{\boldsymbol{f}}_{\boldsymbol{x}} \,. \end{aligned} \tag{15}$$

The series in (9) converges if and only if $\|\boldsymbol{C}\| < 1$, where $\|\cdot\|$ induced norm, or, equivalently, if and only if $\rho(\boldsymbol{C}) < 1$, where $\rho(\cdot)$ spectral radius of the eigenvalues of a matrix. Moreover, if $\rho(\boldsymbol{C}) < 1$, the matrices $\boldsymbol{A}_k$ tend to $\mathbf{0}_{p,p}$ as $k \to \infty$. Hence, based on the definition of $\boldsymbol{A}_k$ in (12), the following condition must hold:

$$\rho(\boldsymbol{C}) = \rho(\boldsymbol{f}_{\boldsymbol{y}}^{-1} \boldsymbol{f}_{\boldsymbol{y}_d}) < 1 \,. \tag{16}$$

which, using the explicit formulation (14), becomes:

$$\rho(\boldsymbol{C}) = \rho(\hat{\boldsymbol{f}}_{\boldsymbol{y}_d}) < 1 \,. \tag{17}$$

The proof of condition (16) is given in Appendix I.

Equation (9) includes a series of infinite terms, which, in actual implementations, has to be truncated at a given value of k (see [24]). In the examples given in the following section, we thus approximate (9) as:

$$\Delta(\lambda) = \lambda \boldsymbol{I}_p - \boldsymbol{A}_0 - e^{-\lambda\tau} \boldsymbol{A}_1 - \sum_{k=2}^{k_m} e^{-\lambda k\tau} \boldsymbol{A}_k \,, \tag{18}$$

where k_m has to be large enough.

The roots of (18) can be calculated in several ways. However, the method based on Chebyshev discretization has proven to be numerically efficient and accurate [23] and will thus be used in the remainder of this paper. The details of the Chebyshev discretization approach can be found in [21]. For clarity, a brief outline of this method is given in Appendix II.

III. Case Studies

This section illustrates the numerical properties of (18) through a variety of physical systems whose dynamic behavior can be described by a NDDEs in the form (1). All cases considered in the remainder of this section are asymptotically stable without delay. The objectives of the numerical appraisal are twofold.

1) To define whether and how the magnitude of the delay τ impacts on the stability of the NDDE. The delay stability margin is then compared with the results of the papers from where the examples discussed in the section were originally proposed.
2) To evaluate the sensitivity of the rightmost eigenvalues with respect to (i) k_m, i.e., the number of matrices $\boldsymbol{A}_k$ in (18); and (ii) N, i.e., the number of points of the Chebyshev discretization grid (see Appendix II). To illustrate the reliability of the simulations, it is important to prove that their results converge with the increase of k_m and N.

With these aims, we consider three examples of NDDEs that are discussed in the literature to show the accuracy and efficiency of the approach.

All simulations are obtained using DOME [25]. The Dome version used for in this case study is based on Python 3.4.2 (http://www.python.org), Nvidia Cuda 7.0, Numpy 1.8.2 (http://numpy.scipy.org), CVXOPT 1.1.8 (http://abel.ee.ucla.edu/cvxopt/), MAGMA 1.6.1 (icl.cs.utk.edu/magma/software), and has been executed on a 64-bit Linux Fedora 21 operating system running on a two Intel Xeon 10 Core 2.2 GHz CPUs, 64 GB of RAM, and a 64-bit NVidia Tesla K20X GPU.

A. Food-limited Population Model

The dynamic food-limited population model introduced in [4] is a scalar nonlinear NDDE in the form of (1):

$$\dot{S}(t) = rS(t)\left[1 - \frac{S(t-\tau) + c\dot{S}(t-\tau)}{K}\right] , \qquad (19)$$

where r and τ are intrinsic growth rate and the recovering time, respectively, of species S, and K is the environment capacity. Parameters r, c, K are positive.

The proposed approach allows evaluating the delay-dependent stability of a stationary solution of (19), i.e., constant population. According to (7), (19) can be rewritten as:

$$\begin{aligned} \dot{x} &= y \qquad (20) \\ 0 &= rx\left(1 - \frac{x_d + cy_d}{K}\right) - y \, . \end{aligned}$$

This model has two equilibrium points, namely $S = 0$ and $S = K$. Given the physical meaning of this model, only the stability of the equilibrium point $S = K$ is of interest.

Reference [27] provides a numerical example of the dynamic bacteria population model (19), with $K = 1$, and

$$r = \frac{\pi}{\sqrt{3}} + \frac{1}{20} , \quad \text{and} \quad c = \frac{\sqrt{3}}{2\pi} - \frac{1}{25} .$$

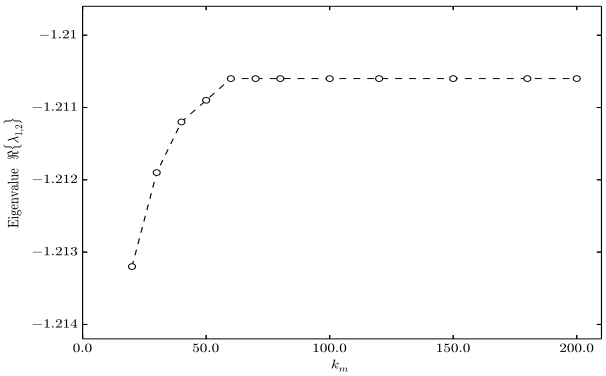

Fig. 1: Rightmost eigenvalue of the dynamic population model (19) as a function of k_m and with $\tau = 0.1$ s and $N = 200$.

TABLE I: Number of points N of the Chebyshev discretization grid to obtain the numerical convergence of the dominant eigenvalues of PEEC (21) with different τ

τ [s]	N	CPU Time [s]
0.0005	100	0.2
0.0008	900	20.8
0.001	4,800	2,453
0.005	6,000	5,460

Figure 1 shows the rightmost eigenvalue's real part as a function of k_m, and assuming $\tau = 0.1$ s and $N = 200$. The rightmost eigenvalue of this model converges for $k_m > 75$. Meanwhile, the imaginary part of the rightmost eigenvalue pair is null.

Figure 2 shows the variations of the rightmost eigenvalue's real part as a function of N, and assuming $\tau = 0.1$ s and $k_m = 100$. The imaginary part of the eigenvalue is not null for $N < 210$. This fact indicates that the Chebyshev discretization introduces spurious eigenvalues when N is not large enough. Also in this case, the numerical analysis is required as N cannot be fixed a priori.

The rightmost eigenvalues of the dynamic population model for unbounded constant delay τ is shown in Fig. 3. The stability boundary is approximately $\tau \approx 14.5$ s, i.e., the quantity of species S is locally asymptotically stable for $\tau < 14.5$ s. Within the stability boundary, the dynamics is characterized by oscillations with relatively low damping. Then for $\tau > 14.5$ s, the unstable operating point give birth to limit cycles. Oscillations disappear, however, for high values of the delay, as both the real and imaginary parts of the dominant eigenvalue go to zero.

B. Linear PEEC Model

In [7], the authors study the numerical solution of a linear NDDE circuit through contractivity analysis, considering PEEC circuit models in the form:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{L}\boldsymbol{x}(t) + \boldsymbol{M}\boldsymbol{x}(t-\tau) + \boldsymbol{N}\dot{\boldsymbol{x}}(t-\tau) , \qquad (21)$$

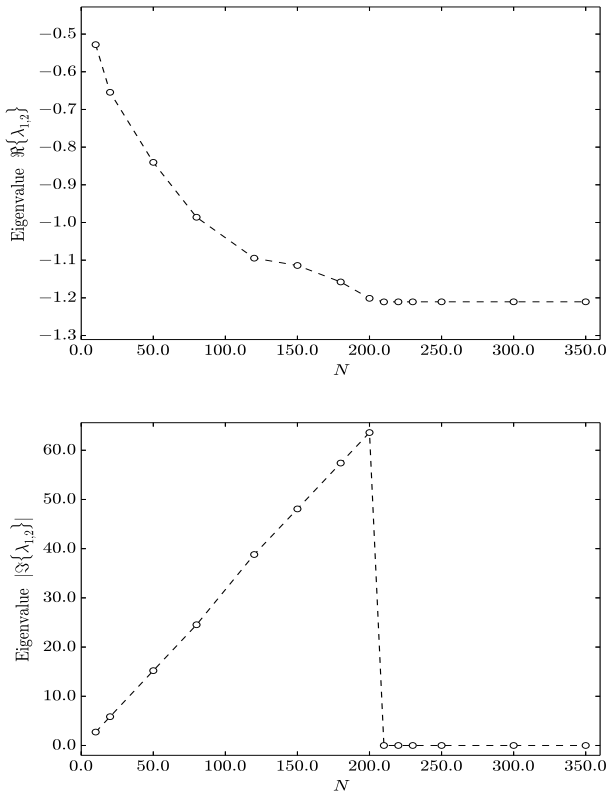

Fig. 2: Rightmost eigenvalue of the dynamic population model (19) as a function of N and with $\tau = 0.1$ and $k_m = 100$.

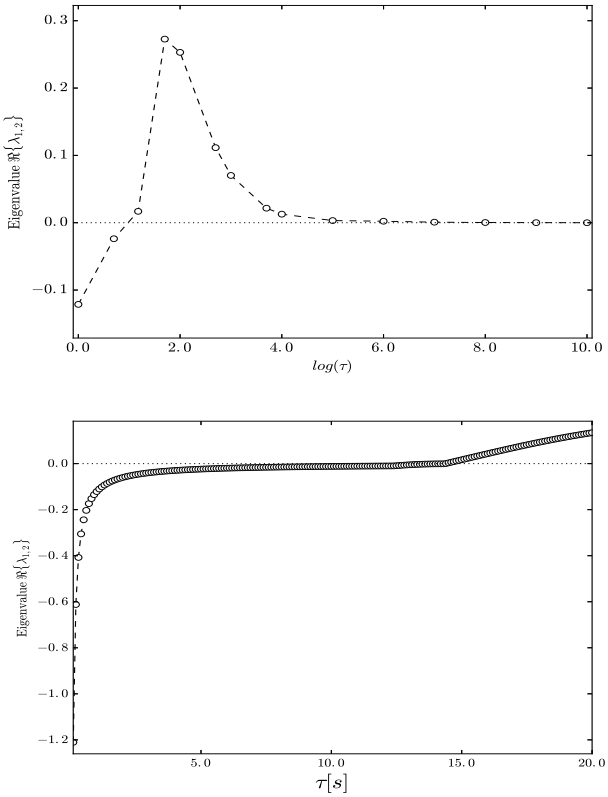

Fig. 3: Real parts of rightmost eigenvalues of the dynamic population model (19) as a function of τ and with $N = 230$ and $k_m = 100$.

In [7], the following parameters are assumed:

$$\frac{\boldsymbol{L}}{100} = \begin{bmatrix} -7 & 1 & 2 \\ 3 & -9 & 0 \\ 1 & 2 & -6 \end{bmatrix}$$

$$\frac{\boldsymbol{M}}{100} = \begin{bmatrix} 1 & 0 & -3 \\ -0.5 & -0.5 & -1 \\ -0.5 & -1.5 & 0 \end{bmatrix}$$

$$\boldsymbol{N} = \frac{1}{72} \begin{bmatrix} -7 & 1 & 2 \\ 3 & -9 & 0 \\ 1 & 2 & -6 \end{bmatrix},$$

With the parameters above, the system (21) is asymptotically stable for $\tau = 1$ s according to the analysis result obtained by contractive continuous Runge-Kutta method. Reference [12] discusses the very same numerical example and indicates that the system is stable for $\tau \leqslant 0.43$ s.

According to the numerical tests based on the descriptor transformed model of (21), the rightmost eigenvalue pair of (21) converges for $k_m \geqslant 20$. The convergence, in this case, is obtained for a relatively small k_m as $\rho(\boldsymbol{N}) = 0.0733 \ll 1$. It is relevant to note that, as τ increases, convergence is obtained at increasingly larger N. The values of N that lead to the convergence of the rightmost eigenvalue pair with different τ are shown in Table I. Note, however, that if N is below the threshold for which the numerical convergence is attained, the results are conservative, as shown in Fig. 4.

The real part of rightmost eigenvalue pairs of (21) with different $\tau \in [0.1, 15]$ s is shown in Fig. 5, using $k_m = 80$, $N = 200$. It can be observed that the delay stability margin is over 15 s, which is much larger than the value identified in [12].

C. Neural Network Model of Neutral Type

This subsection considers the non-linear neutral-type Cohen-Grossberg Neural Network (CGNN) model [8]:

$$\begin{aligned} &\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} + \begin{bmatrix} 0.1 & -0.15 \\ -0.1 & 0.1 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t-\tau) \\ \dot{x}_2(t-\tau) \end{bmatrix} \\ &= \begin{bmatrix} d_1(x_1(t)) & 0 \\ 0 & d_2(x_2(t)) \end{bmatrix} \\ &\times \left(\begin{bmatrix} 12x_1(t) \\ 12x_2(t) \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} f_1(x_1(t)) \\ f_2(x_2(t)) \end{bmatrix} \right. \\ &\left. - \begin{bmatrix} b_{11} & 0.2 \\ b_{21} & -0.125 \end{bmatrix} \begin{bmatrix} f_1(x_1(t-\tau)) \\ f_2(x_2(t-\tau) \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \end{aligned} \tag{22}$$

where $d_i(x_i(t)) = 5 + \sin(x_i(t))$ and $f_i(x_i) = \tanh(x_i)$ for $i = 1, 2$.

In [8], the authors discuss the sufficient delay-independent stability criteria of (22) through the Lyapunov second stability method. According to [8], if $b_{11} = b_{12} = 0.25$, the neural network (22) is delay-independently stable at the unique equilibrium point $\boldsymbol{x}^* = [0.3404, 0.2597]^T$. We can re-obtain the same conclusion using the proposed small-signal stability

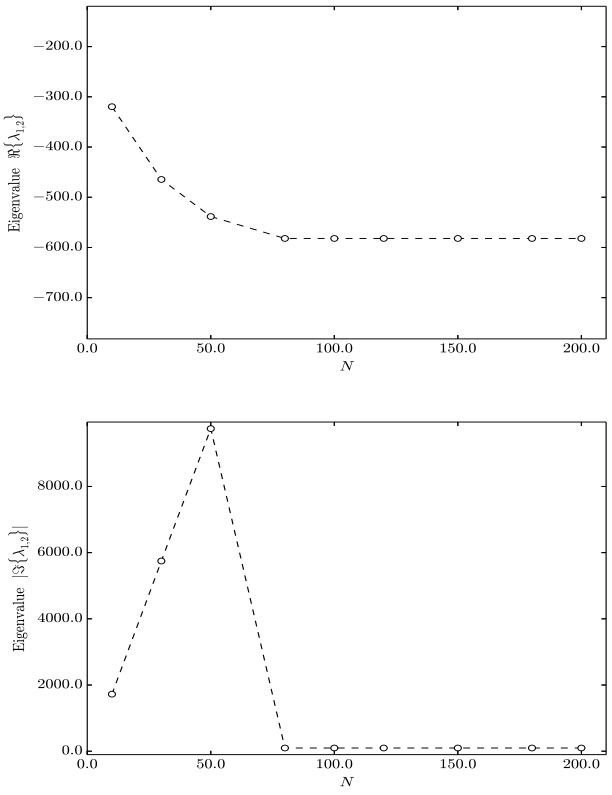

Fig. 4: Rightmost eigenvalues of PEEC (21) as a function of N and with $\tau = 0.0005$ s and $k_m = 80$.

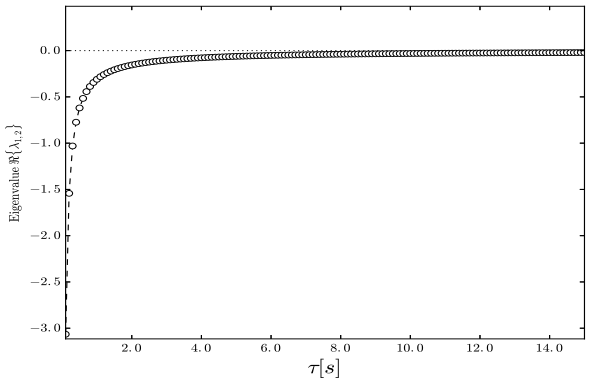

Fig. 5: Rightmost eigenvalues of PEEC (21) as a function of τ and with $k_m = 80$ and $N = 200$.

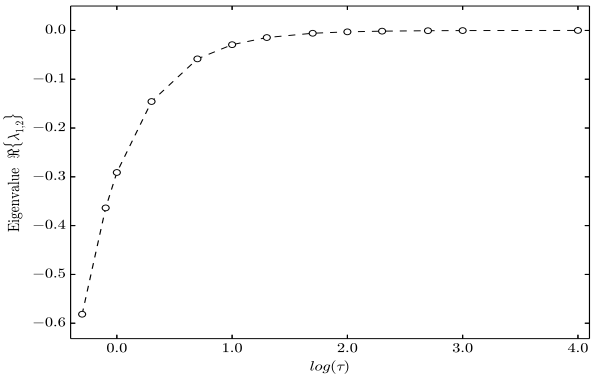

Fig. 6: Rightmost eigenvalues of the neutral-type CGNN (22) as a function of τ and with $k_m = 100$ and $N = 500$

analysis of a DDAE equivalent to (22). Figure 6 shows, in fact, that the dominant eigenvalues of (22) are never positive for $b_{11} = b_{12} = 0.25$ and independently from the value of the delay τ.

IV. Conclusions

The paper provides a derivation of the characteristic equation of NDDEs, i.e., differential equations that include delays in both state variables and their first time derivatives. The characteristic equation is found by means of a descriptor model transformation into an equivalent non-index-1 Hessenberg form DDAE and consists of a series of terms corresponding to infinitely many delays that are multiples of the delays of the original NDDE. The condition for the convergence of the series are also provided in the paper. The paper discusses a numerical appraisals based on a Chebyshev discretization method of the small-signal stability analysis based on a truncated version of the characteristic equation previously determined and defines how the convergence of the series impact on the stability of NDDEs systems. Simulation results indicate that the proposed method allows determining precisely the delay stability margin and, at least for the considered cases, it allows improving the results obtained with other stability analysis methods that are available in the literature.

Appendix I
Convergence of (9)

Next we will prove that the series in (9) converges if and only if (16) holds. For asymptotic stable states we have that $\|\boldsymbol{x}(t-k\tau)\| \leq \mu \in \mathbb{R}$. Then

$$\|\boldsymbol{C}^{k-1} Dx(t-k\tau)\| \leq \|\boldsymbol{C}^{k-1}\|\|D\|\mu$$

The series $\sum_{k=2}^{\infty} C^{k-1}$ converges if and only if $\|\boldsymbol{C}\| < 1$. Hence by using the above inequality and the direct comparison test for series, the sum $\boldsymbol{C}^{k-1} Dx(t-k\tau)$ and consequently the series in (9), converges if and only if $\|\boldsymbol{C}\| < 1$.

Appendix II
Chebyshev discretization scheme

This approach consists in transforming the original problem of computing the roots of (18) into a matrix eigenvalue problem of a PDE system of infinite dimensions. The dimension of the PDE is made tractable using a discretization based on a finite element method. The discretized matrix is build as follows. Let $\boldsymbol{\Xi}_N$ be the Chebyshev discretization matrix of order N (see [21] for details) and define

$$\boldsymbol{M} = \left[\begin{array}{cccc} \hat{\boldsymbol{\Psi}} \otimes \boldsymbol{I}_p & & & \\ \hline \hat{\boldsymbol{A}}_N & \hat{\boldsymbol{A}}_{N-1} & \cdots & \hat{\boldsymbol{A}}_1 \quad \hat{\boldsymbol{A}}_0 \end{array}\right], \qquad (23)$$

where $\otimes$ indicate the *tensor product* or Kronecker product; $\boldsymbol{I}_p$ is the identity matrix of order p; and $\hat{\boldsymbol{\Psi}}$ is a matrix composed of the first $N-1$ rows of $\boldsymbol{\Psi}$ defined as follows:

$$\boldsymbol{\Psi} = -2\boldsymbol{\Xi}_N/\tau \ , \tag{24}$$

and matrices $\hat{\boldsymbol{A}}_0, \ldots, \hat{\boldsymbol{A}}_N$ are defined as follows.

Let consider first the case for which (1) and, hence, (7) include only a single delay τ. Then, equation (18) has k_m delays, with $\tau = \tau_1 < \tau_2 < \cdots < \tau_{k_m-1} < \tau_{k_m} = k_m\tau$. Each point of the Chebyshev grid corresponds to a delay $\theta_j = (N-j)\Delta\tau$, with $j = 1, 2, \ldots, N$ and $\Delta\tau = \tau_{k_m}/(N-1)$. Thus, $j = 1$ corresponds to the state matrix $\boldsymbol{A}_{k_m}$, which corresponds to the maximum delay τ_{k_m}; and $j = N$ is taken by the non-delayed state matrix $\boldsymbol{A}_0$. If a delay $\tau_k = \theta_j$ for some $j = 2, \ldots, N-1$, then the correspondent matrix $\boldsymbol{A}_k$ takes the position j in the grid. For the single-delay case, delays in (18) delays are equally spaced and, hence, this conditions happens if N is a multiple of k_m. However, in general, the delays of the system will not match the points of the grid. Then, a linear interpolation is considered in this paper, as discussed in [23]. The linear interpolation allows also to easily extend the method to the multi-delay case.

Reference [19] shows that the eigenvalues of $\boldsymbol{M}$ are an approximated spectrum of (18). The number of points N of the grid affects the precision and the computational burden of the method, as it is discussed in Section III. In practice, N cannot be very large, as the size of $\boldsymbol{M}$ would prevent applying any numerical technique to compute the eigenvalues.

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