Viable Computation of the Largest Lyapunov Characteristic Exponent for Power Systems

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***Abstract*—Stochastic Differential Algebraic Equations (SDAEs) are used to model power systems. However, there is no universally accepted method to properly evaluate the stability of such models. The theoretical and numerical aspects of the computation of the largest Lyapunov Characteristic Exponent (LCE) for power systems with the inclusion of stochastic processes is discussed as a method to provide a measure of stability. A semi-implicit formulation of power systems is employed in order to exploit parallelism, sparsity and to have low memory requirements. Three case studies are considered, two based on the IEEE 14-bus system as well as a 1,479-bus model of the all island Irish transmission grid.**

***Index Terms*—Lyapunov characteristic exponents (LCEs), stability, limit sets, stochastic processes, differential-algebraic equations (DAEs), deterministic chaos, stochastic chaos.**

I. Introduction

In recent years, power systems have undergone major structural changes due to the desire to promote the use of clean and secure renewable energy. Wind energy is the fastest growing renewable energy source for electricity worldwide which has led to an increase in the volatility and uncertainty present in power systems. As a result, this has incurred drastic changes to how power systems are modeled and simulated. Volatility can be modeled as a set of stochastic processes that, if coupled with dynamics models, lead to describe power systems as a set of SDAEs. There is no universally accepted method to properly evaluate the stability of such models. Hence, the focus of this research is to addresses this issue by proposing a computationally viable method to calculate the largest LCE which can be utilized as a measure of the stability of stochastic power system models.

Modeling power systems as a set of Stochastic Differential Equations (SDEs) was first proposed back in the nineties [1]. However, it is only in recent years that methods to study the impact of stochastic fluctuations on the dynamic response of power systems have started to be developed [2]–[7]. In these papers, the common approach to study the stability of power systems is a Monte Carlo method based on strong solutions of the SDAEs. This requires a high number of time domain simulations, b, which is computationally expensive. The Monte Carlo method is suited to small systems up to a few tens of buses but is not suitable to large systems which yield high dimension SDEs. This paper proposes the application of Lyapunov theory to study the stability of SDE-based power system models which reduces the computational burden from b simulations to 1. The proposed method provides a quantitative solution to outline if a power system is chaotic. It can be used by electrical engineers as an aid to design robust transmission systems as the renewable energy sector grows into the future and is based on the concept of stochastic chaos.

Lyapunov theory was first proposed in [8] and has been widely used in the numerical analysis of nonlinear dynamics since the work of Seydel in [9]. Lyapunov theory is a tool that can be used to capture the sensitive dependence of a system to initial conditions. The LCE provides a measure of how two orbits that start close together converge (or diverge) as time progresses. Roughly speaking, LCEs relate to the trajectories of dynamical systems as eigenvalues relate to equilibrium points. Where eigenvalues indicate the stability of equilibrium points, LCEs indicate the stability of trajectories.

The main difficulty that limits the widespread adoption of LCEs to study chaotic motions and stochastic processes for real world large transmission networks is that such a tool is computationally expensive. In [10] and [11], various methods to compute LCEs are proposed but these methods are limited to linearized models and only small benchmark systems up to a few tens of buses are considered. The calculation of LCEs of real-world nonlinear power system models requires the numerical integration of the variational equation associated with the original dynamic system. Unfortunately, the size of the variational equation increases with the square of the number of state variables of the system. While methods exist for its computation, from a practical point of view, none are applicable to high dimensional systems such as real-world power systems which could have several thousand state and algebraic variables. Herein lies the novelty of this research, to develop a computationally efficient method to viably compute LCEs for power system models of arbitrary size.

The remainder of the paper is organized as follows, in Section II relevant background information of Lyapunov theory and dynamical systems is presented. In Section III, the proposed method is explained along with the mathematics that underpin it. The method discussed in this section is applied to three case studies in Section IV; one deterministic case and two stochastic cases. The computationally efficiency of the proposed approach is illustrated through the calculation of the LCEs of the 1,479-bus model of the all-island Irish grid

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including stochastic wind speed fluctuations. Finally, Section V draws conclusions and outlines possible areas of future work in the field.

II. Lyapunov Theory

This section provides some basic background and definitions of dynamical systems that are relevant for chaotic motions and describes the rationale and practical implementation of LCEs.

Lyapunov exponents are a quantitative method that captures the sensitive dependence of systems to initial conditions. They provide a measure of how two orbits that start close together converge or diverge as time progresses. Diverging orbits are termed chaotic orbits and, in this work, the definition of chaos provided by [12] will be used; a dynamical system is chaotic if it has a positive Lyapunov exponent.

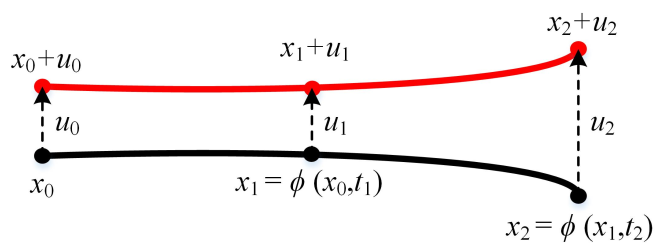

Figure 1: Divergence of two orbits starting from nearby initial points.

Let us consider a set of nonlinear Ordinary Differential Equations (ODEs) in the form:

$$\dot{\boldsymbol{x}} = \boldsymbol{h}(\boldsymbol{x})\,, \tag{1}$$

where $\boldsymbol{x}$, $\boldsymbol{x} \in \mathbb{R}^n$ is the state vector at time t and $\boldsymbol{h}$ is smooth. Consider two nearby points $\boldsymbol{x}_0$ and $\boldsymbol{x}_0+\boldsymbol{u}_0$ in the phase space $\mathcal{M}$, as illustrated in Fig. 1, where $\boldsymbol{u}_0$ is a small perturbation of the initial point $\boldsymbol{x}_0$. After time t, their images under the flow will be $\boldsymbol{\phi}(\boldsymbol{x}_0, t)$ and $\boldsymbol{\phi}(\boldsymbol{x}_0 + \boldsymbol{u}_0, t)$ and the perturbation $\boldsymbol{u}(t)$ is defined as:

$$\boldsymbol{u}(t) = \boldsymbol{\phi}(\boldsymbol{x}_0 + \boldsymbol{u}_0, t) - \boldsymbol{\phi}(\boldsymbol{x}_0, t) = \boldsymbol{\phi}_{\boldsymbol{x}}(\boldsymbol{x}_0, t) \cdot \boldsymbol{u}_0\,, \tag{2}$$

where $\boldsymbol{\phi}_{\boldsymbol{x}}(\boldsymbol{x}_0, t)$ is the gradient of $\boldsymbol{\phi}$ with respect to $\boldsymbol{x}_0$. The average exponential rate of convergence (or divergence) of the two trajectories is defined by:

$$\begin{aligned}\lim_{t\to\infty} e^{\lambda(\boldsymbol{x}_0,\boldsymbol{u}_0)t} &= \lim_{t\to\infty} \frac{\|\boldsymbol{u}(t)\|}{\|\boldsymbol{u}_0\|} \\ &= \lim_{t\to\infty} \|\boldsymbol{\phi}_{\boldsymbol{x}}(\boldsymbol{x}_0, t) \cdot \boldsymbol{u}_0\|\,,\end{aligned}$$

where $\|\boldsymbol{u}\|$ is the norm of the vector $\boldsymbol{u}$. The number λ is termed the LCE and is indicative of the stability of the system.

$$\lambda(\boldsymbol{x}_0, \boldsymbol{u}_0) = \lim_{t\to\infty} \frac{1}{t} \ln \frac{\|\boldsymbol{u}(t)\|}{\|\boldsymbol{u}_0\|} \tag{3}$$

Under weak smoothness conditions on the dynamic system, the limit (3) exists and is finite for almost all points $\boldsymbol{x}_0 \in \mathcal{M}$ and for almost all tangent vectors $\boldsymbol{u}_0$ and it is equal to the largest LCE, λ_1 [13]. If the largest LCE is positive, the nearby orbits exponentially diverge. These are termed chaotic orbits.

For a continuous-time dynamical system (1) and an initial point $\boldsymbol{x}_0$, the tangent vector $\boldsymbol{u}(t)$ defined in (2) evolves in time satisfying the so called *variational equation* [14]:

$$\dot{\boldsymbol{\Phi}}(\boldsymbol{x}_0, t) = \boldsymbol{J}(\boldsymbol{\phi}(\boldsymbol{x}_0, t)) \cdot \boldsymbol{\Phi}(\boldsymbol{x}_0, t), \quad \boldsymbol{\Phi}(\boldsymbol{x}_0, 0) = \boldsymbol{I}_n\,, \tag{4}$$

where $\boldsymbol{\Phi}(\boldsymbol{x}_0, t)$ is the derivative with respect to $\boldsymbol{x}_0$ of $\boldsymbol{\phi}(t)$ at $\boldsymbol{x}_0$ i.e. $\boldsymbol{\Phi}(\boldsymbol{x}_0, t) = \boldsymbol{\phi}_{\boldsymbol{x}}(\boldsymbol{x}_0, t)$ and is termed the *transition matrix*. Equation (4) is a linear time-variant differential equation whose coefficients depend on the evolution of the original system (1). Hence, in general, (4) can be solved only together with (1), as follows:

$$\begin{bmatrix}\dot{\boldsymbol{x}} \\ \dot{\boldsymbol{\Phi}}\end{bmatrix} = \begin{bmatrix}\boldsymbol{h}(\boldsymbol{x}) \\ \boldsymbol{J}(\boldsymbol{x})\end{bmatrix}, \qquad \begin{bmatrix}\boldsymbol{x}(t_0) \\ \boldsymbol{\Phi}(t_0)\end{bmatrix} = \begin{bmatrix}\boldsymbol{x}_0 \\ \boldsymbol{I}_n\end{bmatrix} \tag{5}$$

where $\boldsymbol{J}$ is the Jacobian matrix of $\boldsymbol{h}(\boldsymbol{x})$. The application of traditional integration techniques to the variational equation typically leads to all solutions converging to the largest LCE. In this work, the largest LCE is sufficient to define the asymptotic behavior of a dynamical system. The largest LCE λ_1 can then be estimated by applying (3):

$$\lambda_1 \approx \frac{1}{t} \ln \|\boldsymbol{\Phi}(\boldsymbol{x}_0, t) \cdot \boldsymbol{u}_0\|\,, \tag{6}$$

where $\boldsymbol{u}_0$ is a randomly generated vector of order n. There are three situations to consider for λ_1:

- $\lambda_1 < 0$: The flow is attracted to a stable fixed point or stable periodic orbit.
- $\lambda_1 = 0$: The orbit is a neutral fixed point.
- $\lambda_1 > 0$: The flow is unstable and chaotic.

Negative Lyapunov exponents are characteristic of dissipative or non-conservative systems. Note, bounded infinite trajectories that do not converge towards a fixed point are characterized by at least one zero LCE which corresponds to a perturbation of the phase point along its own trajectory.[1] This certainly happens for SDAEs with bounded trajectories. Hence, the largest LCE satisfies the condition $\lambda_1 \leq 0$ for non-chaotic orbits.

III. LCEs of Differential Algebraic Equations

The determination of the largest LCE is relatively straightforward for ODEs. However, power system models for transient stability analysis cannot be formulated as ODEs. Typically, they are modeled as a set of explicit nonlinear Differential Algebraic Equations (DAEs) [15]. However, in [16], the author proposes a novel semi-implicit model of power systems, that is more general and less computationally demanding than the explicit model. The semi-implicit model is as follows:

$$\begin{bmatrix}\boldsymbol{T} & 0 \\ \boldsymbol{R} & 0\end{bmatrix}\begin{bmatrix}\dot{\boldsymbol{x}} \\ 0\end{bmatrix} = \begin{bmatrix}\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}) \\ \boldsymbol{g}(\boldsymbol{x},\boldsymbol{y})\end{bmatrix}, \tag{7}$$

where $\boldsymbol{x}$, $\boldsymbol{x} \in \mathcal{X} \subset \mathbb{R}^n$ are the state variables; $\boldsymbol{y}$, $\boldsymbol{y} \in \mathcal{Y} \subset \mathbb{R}^m$ are the algebraic variables; $\boldsymbol{f} : \mathcal{X} \times \mathcal{Y} \mapsto \mathbb{R}^n$ is the vector field; and $\boldsymbol{g} : \mathcal{X} \times \mathcal{Y} \mapsto \mathbb{R}^m$ are the algebraic constraints.

[1] Power system models also show an additional vanishing LCE due to the constant motion of the synchronous speed reference which is used to define the rate of change of rotor angles of synchronous machines.

In this paper, for simplicity but without loss of generality, we assume that $\boldsymbol{T}$ and $\boldsymbol{R}$ are time-invariant and very sparse. The time-variant case can be deduced from the developments below. Apart from numerical properties, which are extensively discussed in [16], the main advantage of the formulation in (7) is that $\boldsymbol{T}$ might not be full rank which allows the imposition of an infinitely fast dynamic response to some state variables. This approach greatly reduces the number of operations to compute equations and elements of the Jacobian matrix of the DAE, increases the sparsity of the Jacobian matrix and allows effortless switching between state variables and algebraic variables. The proposed method can be extended to other topologies.

The key difficulty with assessing the LCE of power systems is determining the transition matrix $\boldsymbol{\Phi}(\boldsymbol{x}_0, t)$. Consider a DAE of the form:

$$\boldsymbol{\Gamma}\dot{\boldsymbol{z}}(t) = \boldsymbol{\Psi}(\boldsymbol{z}(t)) \tag{8}$$

where

$$\boldsymbol{\Gamma} = \begin{bmatrix} \boldsymbol{T} & 0 \\ \boldsymbol{R} & 0 \end{bmatrix}, \boldsymbol{z} = \begin{bmatrix} \dot{\boldsymbol{x}} \\ 0 \end{bmatrix} \text{ and } \boldsymbol{\Psi}(z) = \begin{bmatrix} \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) \\ \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}) \end{bmatrix}.$$

The generic solution to (8) is:

$$\boldsymbol{\Gamma}\boldsymbol{z}(t) = \int_0^t \boldsymbol{\Psi}(\boldsymbol{z}(\tau), \tau)d\tau + \boldsymbol{\Gamma}\boldsymbol{z}(0)$$

Expand $\boldsymbol{\Psi}(\boldsymbol{z}(\tau), \tau)$ using the Taylor series expansion about the trajectory $\boldsymbol{\chi}(t; \boldsymbol{\chi}_0)$:

$$\boldsymbol{\Psi}(\boldsymbol{z}(t), t) = \boldsymbol{\Psi}(\boldsymbol{\chi}(t), t) + \left.\frac{\partial \boldsymbol{\Psi}}{\partial \boldsymbol{z}}\right|_{\boldsymbol{z}=\boldsymbol{\chi}(t)} (\boldsymbol{z}(t) - \boldsymbol{\chi}(t)) + \boldsymbol{R}_2(\boldsymbol{z})$$

where $\boldsymbol{R}_2(\boldsymbol{z})$ contains higher order terms that are nonlinear. Only considering first order terms:

$$\boldsymbol{\gamma}(t) = \int_0^t \boldsymbol{\Psi}(\boldsymbol{\chi}(\tau), \tau)d\tau + \int_0^t \boldsymbol{J}(\tau; \boldsymbol{\chi})(\boldsymbol{\gamma}(\tau) - \boldsymbol{\chi}(\tau))d\tau$$

where $\boldsymbol{J}(\tau; \boldsymbol{\chi})$ is the Jacobian matrix of $\boldsymbol{\Psi}$. Letting $\boldsymbol{\zeta}(t) = \boldsymbol{\gamma}(t) - \boldsymbol{\chi}(t)$:

$$\boldsymbol{\Gamma}\boldsymbol{\zeta}(t) = \int_0^t \boldsymbol{J}(\tau; \boldsymbol{\chi})\boldsymbol{\zeta}(t)d\tau + \boldsymbol{\Gamma}\boldsymbol{\zeta}(0) \tag{9}$$

Note that $\boldsymbol{\zeta}(t)$ is the solution of the differential equation:

$$\boldsymbol{\Gamma}\dot{\boldsymbol{\zeta}} = \boldsymbol{J}(t; \boldsymbol{\chi})\boldsymbol{\zeta} \tag{10}$$

The solution to (10) is:

$$\boldsymbol{\zeta}(t) = \boldsymbol{\Phi}(t, 0; \boldsymbol{\chi})\boldsymbol{\zeta}(0) \tag{11}$$

where $\boldsymbol{\Phi}(t, 0; \boldsymbol{\chi})$ is the state transition matrix for (11).

If an implicit integration scheme is utilized, which is highly recommended due to the stiffness of power system models (see, for example, [17] and [18]), the Jacobian matrices required to compute $\boldsymbol{\Phi}$ are already available because these are needed to integrate the DAE. Moreover, since the variational equation is linear, one can implement the explicit expression to integrate such an equation. For example, using the implicit backward Euler method with a time step of Δt, (10) becomes:

$$\boldsymbol{\Gamma}\hat{\boldsymbol{\zeta}}(\Delta t) = \hat{\boldsymbol{\zeta}}(0) + \Delta t \boldsymbol{J}(\hat{\boldsymbol{\chi}}(\Delta t))\hat{\boldsymbol{\zeta}}(\Delta t)$$

$$\hat{\boldsymbol{\zeta}}(\Delta t) = [\boldsymbol{\Gamma} - \Delta t \boldsymbol{J}(\hat{\boldsymbol{\chi}}(\Delta t))]^{-1}\hat{\boldsymbol{\zeta}}(0) \tag{12}$$

and

$$\hat{\boldsymbol{\zeta}}(k\Delta t) = \prod_{i=1}^{k} [\boldsymbol{\Gamma} - \Delta t \boldsymbol{J}(\hat{\boldsymbol{\chi}}((k-i+1)\Delta t))]^{-1}\hat{\boldsymbol{\zeta}}(0) \tag{13}$$

Thus, the state transition matrix in (11) is therefore given by:

$$\boldsymbol{\Phi}(k\Delta t, 0; \boldsymbol{\chi}_0) = \prod_{i=1}^{k} [\boldsymbol{\Gamma} - \Delta t \boldsymbol{J}(\hat{\boldsymbol{\chi}}((k-i+1)\Delta t))]^{-1} \tag{14}$$

In the next section, we will demonstrate the computational viability of the proposed approach to determine the transition matrix and the largest LCE for power systems.

IV. Case Studies

This section presents three case studies. Subsection IV-A focuses on the IEEE 14-bus system, which is known to show a deterministic chaotic behavior for sufficiently high loading levels [19]. The IEEE 14-bus system with the inclusion of noise in the load power consumption is considered in Subsection IV-B to define the behavior of LCEs for non-deterministic chaotic motions. Subsection IV-C considers the all-island Irish transmission system. This is a large real-world network where the computational burden of the variational equation would typically restrict the calculation of LCEs. The computational viability of the proposed method is outlined here.

All simulations are obtained using Dome, a Python-based power system software tool [20]. The Dome version utilized in this case study is based on Python 3.4.3; ATLAS 3.10.2 for dense vector and matrix operations; CVXOPT 1.1.8 for sparse matrix operations; and KLU 1.3.6 – included in SuiteSparse 4.5.1 – for sparse matrix factorization. All simulations were executed on a 64-bit Ubuntu 14.04 operating system running on a 8 core 3.60 GHz Intel Xeon with 12 GB of RAM.

A. IEEE 14-bus System - Deterministic Chaos

The IEEE benchmark network consists of 2 synchronous machines and 3 synchronous compensators, 2 two-winding and 1 three-winding transformers, 15 transmission lines and 11 loads. The system also includes primary voltage regulators (AVRs) and a PSS connected at machine 1. All data of the IEEE 14-bus system as well as a detailed discussion of its transient behavior can be found in [15].

The system includes 63 state variables, of which only 52 are associated to non-null time constants, and 84 algebraic variables. Null-time constants are associated to synchronous machine stator fluxes, ψ_{sd} and ψ_{sq}, and the variable e'_d whose time constant, T'_{q0}, is null for the machine connected at bus 1. Hence in (7) $n = 63$ and $m = 84$.

In [19], the authors show that increasing the loading level of the IEEE 14-bus system without PSS leads to a Hopf bifurcation followed by a series of period-doubling bifurcations

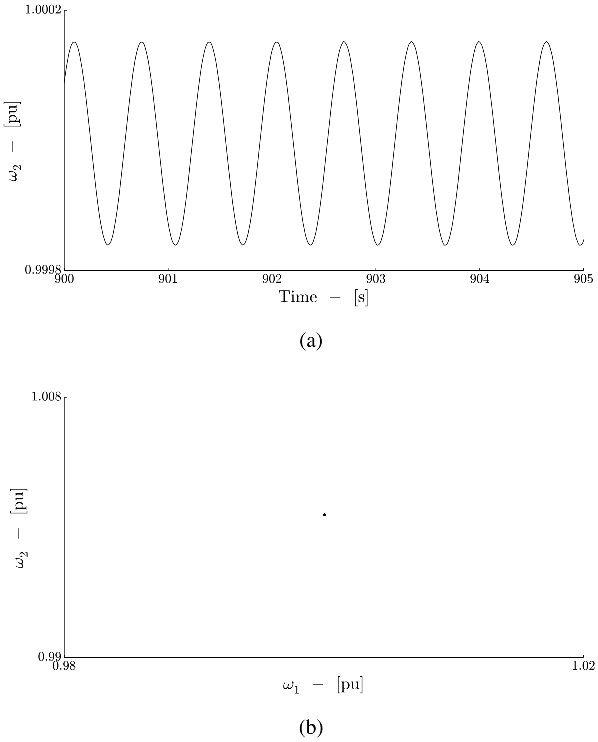

Figure 2: (a) Synchronous generator 2 speed (b) Poincaré section for ω_1 and ω_2 for $K_\omega = 0.5$.

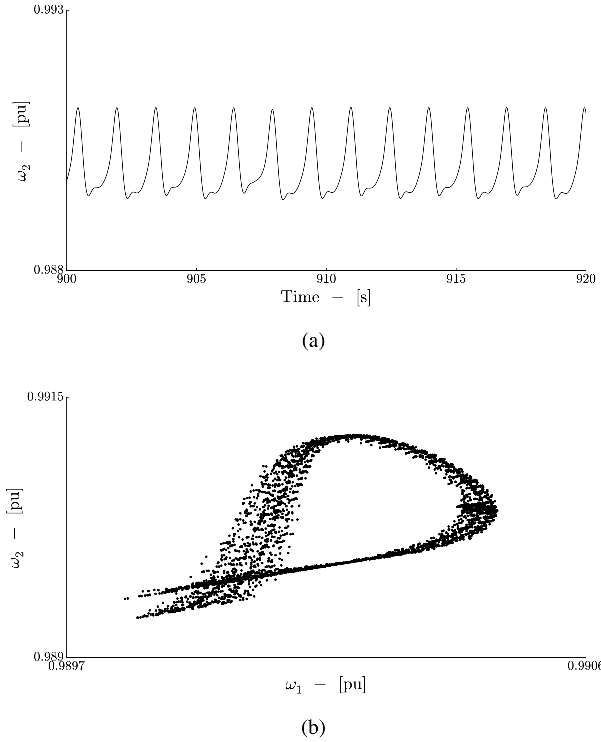

Figure 3: (a) Synchronous generator 2 speed (b) Poincaré section for ω_1 and ω_2 for $K_\omega = 50$.

that eventually leads to the appearance of chaos. It is also shown that chaos can be eliminated by including the PSS if a proper PSS model with an accurate tuning of the parameters is implemented. However, if the PSS is not properly tuned, it is possible that the PSS may trigger a series of Hopf bifurcations leading to chaos.

Consider the case when $K_\omega = 0.5$, the PSS is not properly tuned and the system is oscillating, Fig. 2 (a). It is clear that the output is a limit cycle with a period of $T \approx 0.653$ (s). One conventional method to assess the behavior of oscillating systems is a Poincaré section; two state variables are sampled at a frequency f and plotted against one another. Figure 2 (b) shows the Poincaré section where ω_1 and ω_2, rotor speeds 1 and 2, are sampled at $f = 1/0.653$. Since the system oscillates with the same period as the sampling frequency a single point is visible on the Poincaré section which is indicative of a stable limit-cycle i.e. the system is not chaotic.

Now consider the case when $K_\omega = 50$, the PSS is not properly tuned and the system is wildly oscillating, Fig. 3 (a). Unlike Fig. 2 (a), it is difficult to visually determine whether the output is periodic or aperiodic. However, the Poincaré section reveals that the output is aperiodic as output generates a fractal pattern on the Poincaré map which is indicative of chaos. In this instance, it cannot be determined if the orbit is chaotic or a quasi-periodic; the motion is associated with a finite number of frequencies that are related to one another by irrational multiples.

While Poincaré sections are useful to determine the type of behavior of deterministic systems operate with, they offer little insight into the stability of higher order orbits. We will now consider the largest LCEs for both K_ω values.

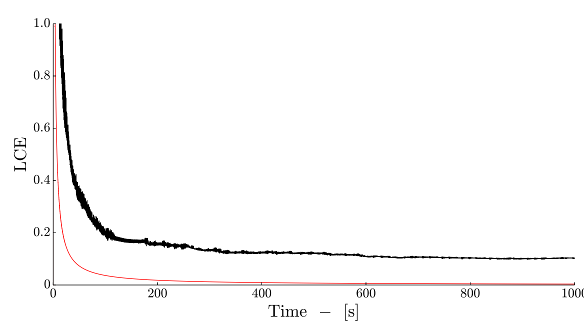

Figure 4: Largest LCE of the IEEE 14-bus system with $K_\omega = 0.5$ (red) and $K_\omega = 50$ (black)

Figure 4 shows the largest LCE for the IEEE 14-bus system for the two values of K_ω. In red, with $K_\omega = 0.5$, the system operates with a limit cycle with a period of $T = 0.6533$ (s), the largest LCE settles to 0. This is the expected value for the largest LCE as the system is not operating in the chaotic region. However, when $K_\omega = 50$, the largest LCE settles to 0.2 i.e. a positive number which is an indicator of chaos. Unlike the Poincaré section, this quantitative measure acts as a binary measure as to whether the system is chaotic or not. When stochastic systems are considered, qualitative measures are not applicable as plots will appear random and thus, chaotic. There is no way to qualitatively determine if these systems are chaotic.

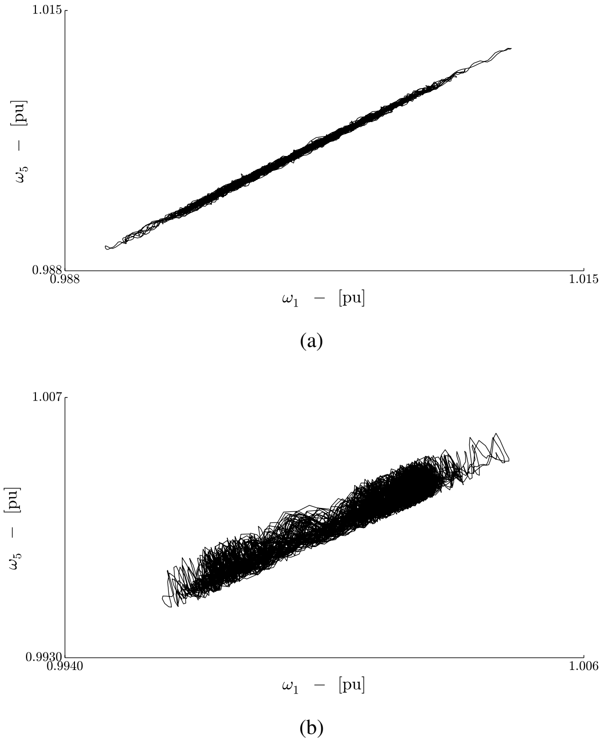

Figure 5: Flow of the state space of rotor speeds of machines 1 and 5 for (a) scenario 1 and (b) scenario 2.

The CPU simulations time to calculate the largest LCE is 5 (s) with $K_\omega = 50$. The total simulation time is 54 (s). Therefore, the calculation of the LCE increased the total simulation time by approximately 10 %. The simulations used a fixed integration step of $\Delta t = 0.01$ (s). However, this is a small benchmark system with no stochastic perturbations. We will now consider the effect of stochastic perturbations on the computational burden of the LCE.

B. IEEE 14-bus System - Stochastic Chaos

This subsection discusses the effect of noise on the dynamic behavior of the IEEE 14-bus system. Noise is modeled as Ornstein-Uhlenbeck (OU) process as discussed in [6]. This is a Wiener process characterized by a normal distribution, exponential autocorrelation and constant standard deviation. The parameters of the OU processes of the loads are the same as those considered in [7]. The following scenarios are considered.

- Original system with 5 synchronous machines and two primary frequency regulators at generators 1 and 2 (see Fig. 5 (a)).
- A wind power plant substituting the synchronous generator at bus 2; only the generator at bus 1 includes primary frequency regulation (see Fig. 5 (b)).

Figure 5 illustrates the flow of the state space of rotor speeds 1 and 5 for both scenarios. Unlike the Poincaré sections in Section IV-A, these diagrams offer little insight into the stability of the system. Since the system is operating with stochastic loads, the flow in the state space appears to be random and therefore has the appearance of chaos. There is no method to select a sampling frequency and to plot a Poincaré section so as to filter out the stochastic perturbations. As a result, conventional qualitative deterministic methods are not applicable to stochastic situations. In both scenarios, the trajectories of the machine rotor speeds appear to be similar. There are no major differences. For this reason, the largest LCE is proposed as a method to study the stability of stochastic power systems as it is a quantitative method and not open to subjective interpretation. This provides a method to distinguish between the two modes of operation

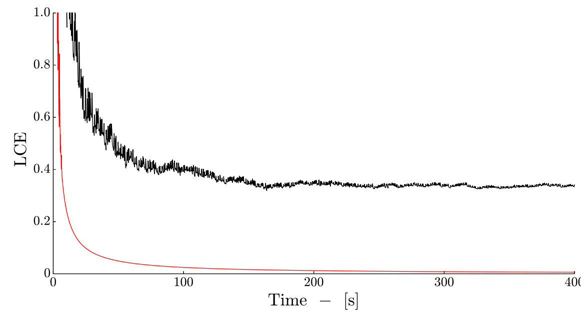

Figure 6: Largest LCE of the IEEE 14-bus system for (red) scenario 1 and (black) scenario 2.

Figure 6 shows the largest LCE for both stochastic situations. With all the synchronous machines connected, the largest LCE settles to 0 and indicates a stable system. For the second scenario, when wind is introduced, the system has a positive LCE and is unstable. It is to be expected that the more regulation that is present in a system, the more robust a system is. The evaluation of the LCEs provides a quantitative measure of such robustness. Thus, subjective interpretation using qualitative approaches are not required. This section demonstrates the promising tool of LCEs to define the impact of stochastic processes on the performance of power system controllers.

C. All-Island Irish Transmission System

In this final case study, the all-island Irish transmission system set up at the UCD Electricity Research Center is considered. The model includes $1,479$ buses, $1,851$ transmission lines and transformers, 245 loads, 22 conventional synchronous power plants with AVRs and turbine governors, 6 PSSs and 176 wind power plants. The topology and the data of the transmission system are based on the real-world system provided by the Irish TSO, EirGrid. However, dynamic data is estimated and is based on the authors knowledge of the technology of power plants. Hence, simulation results, while realistic, do not represents actual operating condition of the Irish transmission system.

The main purpose of this section is to highlight the computational viability of the proposed method for computing the largest LCE for a large system. The system includes $2,112$ state variables, of which $2,064$ have non-null time constants, and $6,338$ algebraic variables.

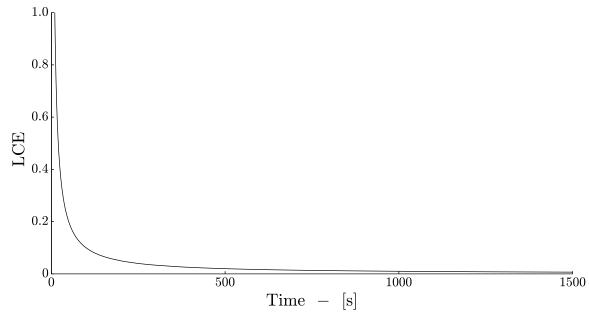

Figure 7: Largest LCE for the all-island Irish system.

The system is simulated for 1500 (s) with a time step of 0.01 (s). The simulation takes 4 minutes and 35 seconds without the calculation of the variational equation and the largest LCE. Figure 7 shows the largest LCE is stable for the Irish transmission system. The calculation of the largest LCE takes an additional 5 minutes and 42 seconds. This approximately doubles the simulation time for the system.

This section has highlighted the feasibility of using LCEs to quantify chaos in large power systems of arbitrary size. The main advantages of the proposed approach compared to Monte Carlo based methods are as follows:

- **Computationally efficient**: this work has demonstrated that the proposed method is computationally viable and efficient for power systems of arbitrary size. Monte Carlo based methods require a high number of simulations to be performed for statistical accuracy. The calculation of the largest LCE requires one.
- **Binary result**: the largest LCE is a single number which can be used to determine whether the system is stable or chaotic.
- **Robust**: Monte Carlo based methods rely on statistical accuracy which does not guarantee that all possible dynamics are captured even if b is high. Lyapunov theory does not suffer from the same constraint.

However, it is important to note that the largest LCE is limited in its approach. Monte Carlo based methods can also be used as a qualitative tool to assess whether power systems are chaotic. They have one key advantage compared to the largest LCE, they can provide other insights into power systems such as the statistical information outlined in references [2]–[7]. The largest LCE gives a measure of the stability of the system but requires further simulations to gain more insights.

V. Conclusions

This paper highlights the key issue stopping the widespread adoption of the largest LCE as a stability measure for both deterministic and stochastic chaotic motions in power systems as the computational burden associated with the variational equation for real-world power systems. The size variational equation is equal to the square of the number of state variables in the system. Historically, this has been a prohibitive size as the computational burden is too great to warrant its inclusion.

The main contribution of this paper is to provide a computationally viable method to estimate the largest LCE for power systems through the use of semi-implicit SDAEs, by proposing a method to efficiently calculate the transition matrix. This contribution was highlighted using three case studies to study the calculation of LCEs for both deterministic and stochastic power systems as well as demonstrating its effectiveness for large real world power systems through the all-island Irish transmission system.

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