Construction of SDE-based wind speed models with exponential autocorrelation

Rafael Zárate Miñano

Escuela de Ingeniería Minera e Industrial de Almadén

Universidad de Castilla-La Mancha

Introduction
Motivation

- Use of wind speed models in the analysis of many aspects of power systems
  - power system economics and operation
  - generation capacity reliability evaluation
  - dynamic studies and control of wind turbines
- Different types of models
  - time series models
  - four-component composite models
  - models based on Kalman filters
  - based on stochastic differential equations

Introduction
Motivation

- The wind speed must be properly characterized since the reliability of the different studies depends on it

- Statistical characterization:
  - Probability distribution: Weibull, Gamma, etc ...
  - Autocorrelation: Exponential or Power-law

- Models proposed in the literature fail in reproducing one of the above characteristics

Introduction
Contribution

- We relay on basic stochastic calculus concepts and tools ...
    - Stationary Markov processes
    - Regression theorem
    - Itô formula
    - Fokker-Planck equation
- ... to derive a method to construct SDEs that exactly reproduce both the probability distribution and the exponential autocorrelation of the wind speed

Outlines of Stochastic Calculus
Stationary Markov processes

- Markov process: Stochastic process without memory
  - The future of the process only depends on the present but it is independent on the past
- Stationary process: The probability distribution is time-invariant
  - $E[x(t)] = \mu(t) = \mu$
  - $\sqrt{E[(x(t) - \mu)^2]} = \sigma(t) = \sigma$

Outlines of Stochastic Calculus
Stationary Markov processes

- For autocovariance and autocorrelation, stationarity implies:

$$c(s,t) = E\left[(x(s) - \mu(s)) \cdot (x(t) - \mu(t))\right]$$

$$r(s,t) = \frac{E\left[(x(s) - \mu(s)) \cdot (x(t) - \mu(t))\right]}{\sigma(s) \cdot \sigma(t)}$$

depend only on the time lag $\tau = t - s$, that is

$$c(s,t) = c(\tau) = E\left[(x(t-\tau) - \mu) \cdot (x(t) - \mu)\right]$$

$$r(s,t) = r(\tau) = \frac{E\left[(x(t-\tau) - \mu) \cdot (x(t) - \mu)\right]}{\sigma^2}$$

Outlines of Stochastic Calculus
Regression theorem

$$\frac{dE[x(t)]}{dt} = -\alpha \cdot E[x(t)]$$

$$\Downarrow$$

$$\frac{dc(\tau)}{d\tau} = -\alpha \cdot c(\tau)$$

$$\Downarrow$$

Autocovariance $\rightarrow$ $c(\tau) = \sigma^2 \cdot e^{-\alpha \cdot \tau}$

Autocorrelation $\rightarrow$ $r(\tau) = e^{-\alpha \cdot \tau}$

Outlines of Stochastic Calculus
Stochastic differential equations

- Differential form:

$$dx(t) = a(x(t), t) \cdot dt + b(x(t), t) \cdot dW(t)$$

- Integral form:

$$x(t) - x_0 = \int_0^t a(x(u), u) \cdot du + \int_0^t b(x(u), s) \cdot dW(u)$$

Outlines of Stochastic Calculus
The Itô formula

$$dg(x(t),t) =$$

$$\left[\frac{\partial g(x(t),t)}{\partial t} + a(x(t),t)\cdot\frac{\partial g(x(t),t)}{\partial x(t)}\right.$$

$$\left. + \frac{1}{2}\cdot b^2(x(t),t)\cdot\frac{\partial^2 g(x(t),t)}{\partial x^2(t)}\right]\cdot dt$$

$$+ b(x(t),t)\cdot\frac{\partial g(x(t),t)}{\partial x(t)}\cdot dW(t)$$

Outlines of Stochastic Calculus
Fokker-Planck equation

$$\frac{\partial p(x(t),t)}{\partial t} =$$

$$-\frac{\partial}{\partial x(t)}\left[a(x(t),t)\cdot p(x(t),t)\right]$$

$$+\frac{1}{2}\cdot\frac{\partial^2}{\partial x^2(t)}\left[b^2(x(t),t)\cdot p(x(t),t)\right]$$

Proposed Building Method of the SDE Model
Use of the Fokker-Planck equation

- For stationary process:

$$a(x(t),t) = a(x(t))$$
$$b(x(t),t) = b(x(t))$$
$$p(x(t),t) = p(x(t))$$

- Fokker-Planck equation reduces to:

$$0 = -a(x(t)) \cdot p(x(t)) + \frac{1}{2} \cdot \frac{\partial}{\partial x(t)} \left[b^2(x(t)) \cdot p(x(t)) \right]$$

Proposed Building Method of the SDE Model

Use of the Fokker-Planck equation

- Solving the Fokker-Planck equation for $a(x(t))$:

$$a(x(t)) = b(x(t)) \cdot \frac{\partial b(x(t))}{\partial x(t)} + \frac{1}{2} \cdot b^2(x(t)) \cdot \frac{\partial \ln p(x(t))}{\partial x(t)}$$

- Solving the Fokker-Planck equation for $b^2(x(t))$:

$$b^2(x(t)) = \frac{2}{p(x(t))} \cdot \int_{-\infty}^{x(t)} a(z(t)) \cdot p(z(t)) \cdot dz(t)$$

Proposed Building Method of the SDE Model

Use of the Itô formula

- We look for a differential equation for the autocovariance:

$$g(x(t)) = (x(t) - \mu) \cdot (x(s) - \mu), \quad \text{where } \; s < t$$

- Applying the Itô formula:

$$\frac{\partial g(x(t))}{\partial t} = 0$$

$$\frac{\partial g(x(t))}{\partial x(t)} = x(s) - \mu$$

$$\frac{\partial^2 g(x(t))}{\partial x^2(t)} = 0$$

Proposed Building Method of the SDE Model

Use of the Itô formula

- Result:

$$d[(x(t)-\mu)\cdot(x(s)-\mu)] = \\ a(x(t))\cdot(x(s)-\mu)\cdot dt + b(x(t))\cdot(x(s)-\mu)\cdot dW(t)$$

with initial condition $(x(s)-\mu)^2$.

- Integral form:

$$(x(t)-\mu)\cdot(x(s)-\mu)-(x(s)-\mu)^2 = \\ \int_s^t a(x(u))\cdot(x(s)-\mu)\cdot du + \int_s^t b(x(u))\cdot(x(s)-\mu)\cdot dW(u)$$

Proposed Building Method of the SDE Model
Use of the Itô formula

- Applying the expectation operator:

$$E[(x(t)-\mu)\cdot(x(s)-\mu)] - E[(x(s)-\mu)^2] =$$
$$E\left[\int_s^t a(x(u))\cdot(x(s)-\mu)\cdot du\right]$$
$$+E\left[\int_s^t b(x(u))\cdot(x(s)-\mu)\cdot dW(u)\right]$$

and taking into account that:

$$E\left[\int f(x(t))\cdot dW(t)\right] = 0$$

Proposed Building Method of the SDE Model

Use of the Itô formula

$$E\left[(x(t)-\mu)\cdot(x(s)-\mu)\right]-E\left[(x(s)-\mu)^2\right]=\int_s^t E\left[a(x(u))\cdot(x(s)-\mu)\right]\cdot du$$

- Coming back to the differential form:

$$\frac{dE\left[(x(t)-\mu)\cdot(x(s)-\mu)\right]}{dt}=E\left[a(x(t))\cdot(x(s)-\mu)\right]$$

Proposed Building Method of the SDE Model

Use of the Itô formula

- By comparing

$$\frac{dE\left[(x(t)-\mu)\cdot(x(s)-\mu)\right]}{dt} = E\left[a(x(t))\cdot(x(s)-\mu)\right]$$

with

$$\frac{dc(s,t)}{dt} = -\alpha \cdot c(s,t)$$

where

$$c(s,t) = E\left[(x(t)-\mu)\cdot(x(s)-\mu)\right]$$

it is clear that

$$\boxed{a(x(t)) = -\alpha \cdot (x(t)-\mu)}$$

Proposed Building Method of the SDE Model
Summary

- To have a wind speed model with exponential autocorrelation:

1. Perform a statistical analysis of the wind speed data
   - Identify the probability distribution $p(x(t))$
   - Identify the autocorrelation coefficient α

Proposed Building Method of the SDE Model
Summary

2 Use a SDE of the form:

$$dx(t) = a(x(t)) \cdot dt + b(x(t)) \cdot dW(t)$$

- The drift term is:

$$a(x(t)) = -\alpha \cdot (x(t) - \mu)$$

- The diffusion term is computed from:

$$b^2(x(t)) = \frac{2}{p(x(t))} \cdot \int_{-\infty}^{x(t)} -\alpha \cdot (z(t) - \mu) \cdot p(z(t)) \cdot dz(t)$$

Examples

Three-parameter Beta distribution

$$p_{\mathrm{B}}(x) = \begin{cases} \dfrac{1}{\lambda_3 \cdot B\left(\lambda_1, \lambda_2\right)} \cdot \left(\dfrac{x}{\lambda_3}\right)^{\lambda_1 - 1} \cdot \left(\dfrac{\lambda_3 - x}{\lambda_3}\right)^{\lambda_2 - 1} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

$$a(x) = -\alpha \cdot \left(x - \frac{\lambda_1 \cdot \lambda_3}{\lambda_1 + \lambda_2}\right)$$

$$b(x) = \sqrt{\frac{2 \cdot \alpha \cdot (\lambda_3 - x) \cdot x}{\lambda_1 + \lambda_2}}$$

Examples

Two-parameter Gamma distribution

$$p_{\mathrm{G}}(x) = \begin{cases} \dfrac{1}{{\lambda_2}^{\lambda_1} \cdot \Gamma\left(\lambda_1\right)} \cdot x^{\lambda_1 - 1} \cdot \exp\left(-\dfrac{x}{\lambda_2}\right) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

$$a(x) = -\alpha \cdot (x - \lambda_1 \cdot \lambda_2)$$

$$b(x) = \sqrt{2 \cdot \alpha \cdot \lambda_2 \cdot x}$$

Examples
Two-parameter Weibull distribution

$$p_{\mathrm{W}}(x) = \begin{cases} \frac{\lambda_1}{\lambda_2} \cdot \left(\frac{x}{\lambda_2}\right)^{\lambda_1 - 1} \cdot \exp\left(-\left(\frac{x}{\lambda_2}\right)^{\lambda_1}\right) & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

$$a(x) = -\alpha \cdot \left(x - \lambda_2 \cdot \Gamma\left(1 + \frac{1}{\lambda_1}\right)\right)$$

$$b(x) = \sqrt{b_1(x) \cdot b_2(x)}$$

Examples
Two-parameter Weibull distribution

$$b_1(x) = 2 \cdot \alpha \cdot \frac{\lambda_2}{{\lambda_1}^2} \cdot x \cdot \left(\frac{\lambda_2}{x}\right)^{\lambda_1}$$

$$b_2(x) = \lambda_1 \cdot \exp\left(\left(\frac{x}{\lambda_2}\right)^{\lambda_1}\right) \cdot \Gamma\left(1 + \frac{1}{\lambda_1}, \left(\frac{x}{\lambda_2}\right)^{\lambda_1}\right) - \Gamma\left(\frac{1}{\lambda_1}\right)$$

Numerical Simulations
Three-parameter Beta distribution

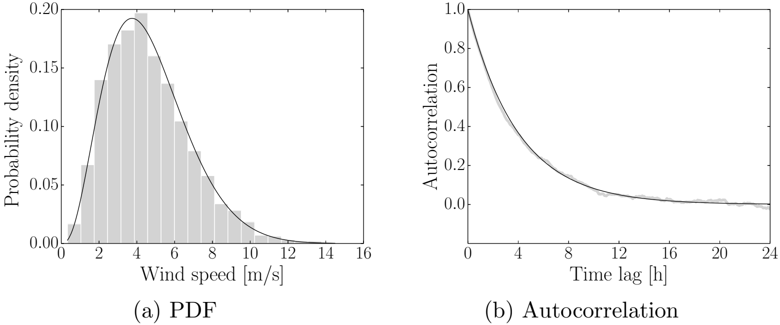
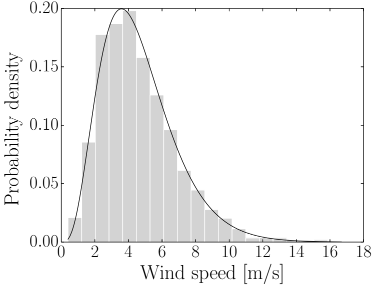
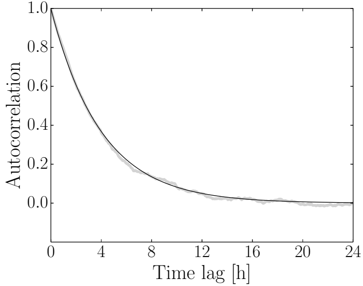


Figure: Three-parameter Beta distribution model.

Numerical Simulations
Two-parameter Gamma distribution



(a) PDF



(b) Autocorrelation

Figure: Two-parameter Gamma distribution model.

Numerical Simulations
Two-parameter Weibull distribution

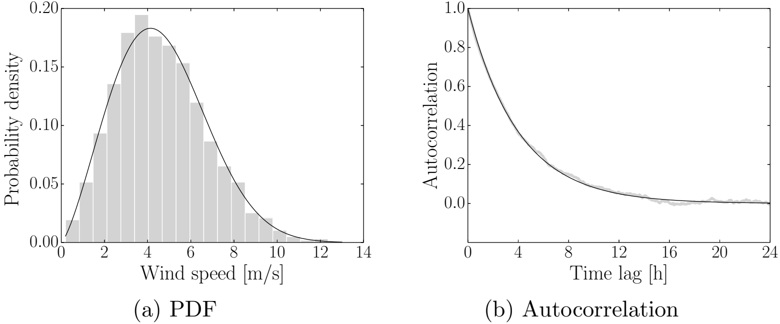


Figure: Two-parameter Weibull distribution model.

Case Study

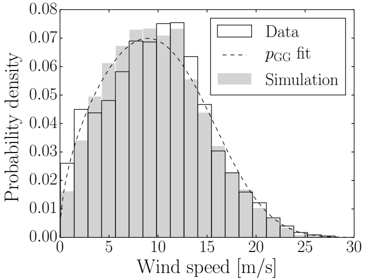
- Modeling the wind speed in Wellington, New Zealand

- Data set: hourly-mean values for whole year 2014 (8760 values)

Case Study

p_{B}	26520.45
p_{G}	26934.30
p_{GG}	26405.15
p_{IG}	29302.87
p_{LN}	27953.50
p_{R}	26548.48
p_{TN}	27439.81
p_{W}	26546.42

(a)



(b)

Figure: (a) Negative log likelihood value of the PDFs parameter estimation; (b) Generalized Gamma PDF fit to the data histogram and histogram of the simulated process.

Case Study

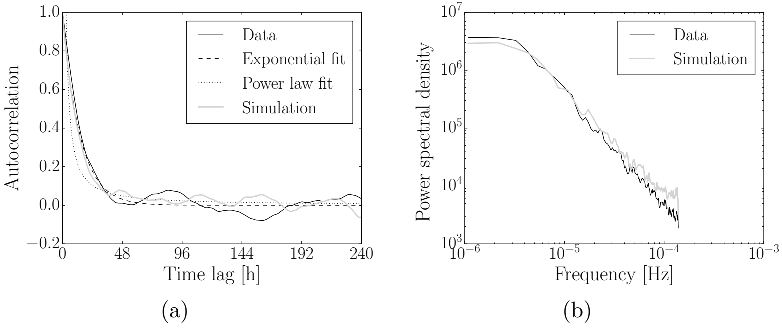


Figure: (a) Autocorrelation analysis of data and autocorrelation of the simulated process; (b) Power spectral density of data and of the simulated process.