ENGINEERING APPLICATIONS OF ELECTRIC SYSTEMS SIMULATION
A primer on switching electrical circuit simulation

Federico Bizzarri

Dipartimento di Elettronica e Informazione, Politecnico di Milano
P.za Leonardo da Vinci 32, I-20133 Milano, Italy
e-mail: federico.bizzarri@polimi.it

Summary

something about what one should know when dealing with circuits (and systems) described by a <u>smooth vector field</u> and working at <u>periodic steady state</u>

- Modified Nodal Analysis Formulation (this is for circuits)
- Periodic Steady State Analysis (time-domain shooting method)
- Periodic small signal analysis
- Floquet Theory (stability)

Summary

extension to circuits described by a "non-smooth" vector field

- Hybrid dynamical systems (in a nutshell)
- Saltation matrix
  - Details in the switching scenario
  - General formulation
- A simple case study

Summary

something about what one should know when dealing with circuits (and systems) described by a smooth vector field working at periodic steady state

- **Modified Nodal Analysis Formulation (this is for circuits)**
- Periodic Steady State Analysis (time-domain shooting method)
- Periodic small signal analysis
- Floquet Theory (stability)

Modified nodal analysis (MNA)

1. Modified nodal analysis allows to derive the mathematical model of an electric dynamical circuit described by
    - topological equations (Kirchhoff's voltage and current laws) and
    - constitutive equations of the components making up the circuit itself.

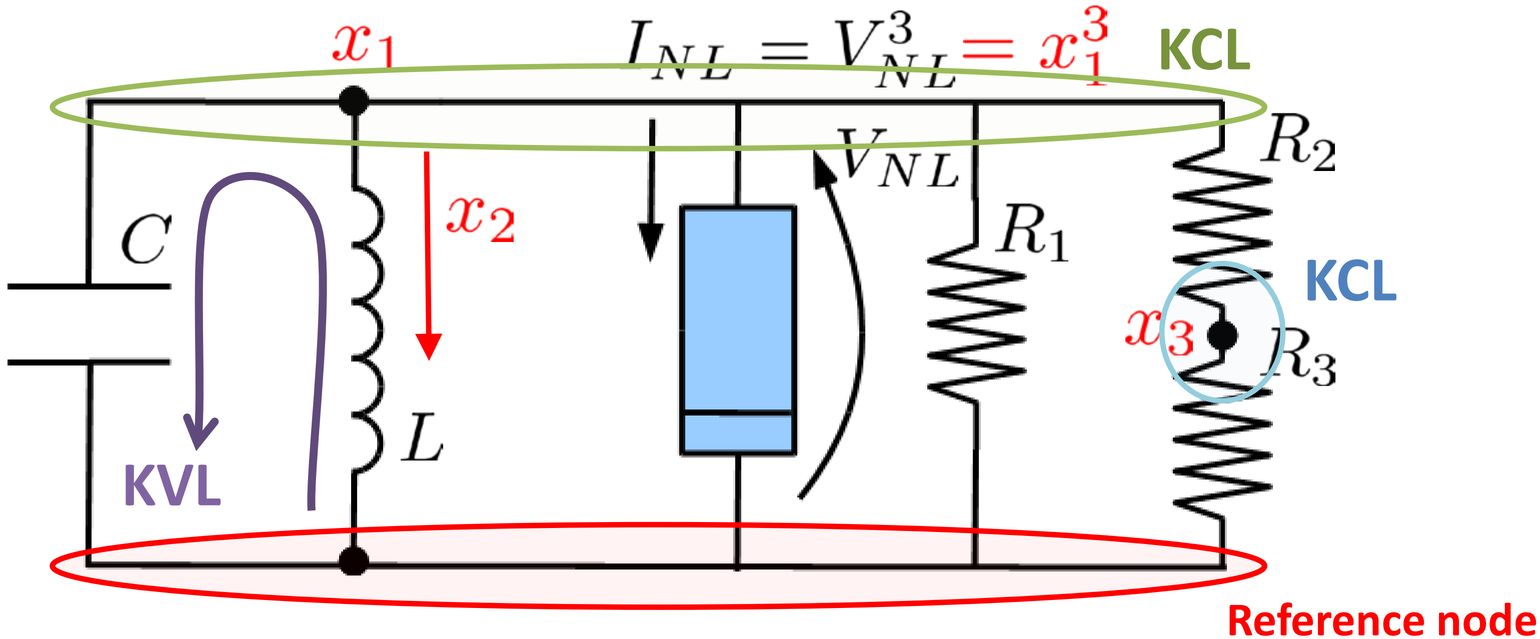
   This model can be profitably used to describe the circuit behavior.

2. The unknowns of the MNA model are the circuit node potentials (with respect to a reference node) and the edge currents of both those components that are not voltage controlled (e.g., independent voltage source) or that have current as a state variable (e.g., inductor).

3. The MNA model of non linear dynamical circuits is a set of Differential and Algebraic Equations constituting a DAE. It becomes simply algebraic for non-dynamical (resistive) networks

4. **MNA is crucial to systematically described an electric circuit and to study its behavior from a numerical stand point (viz., through a circuit simulator).**

Modified nodal analysis (MNA)

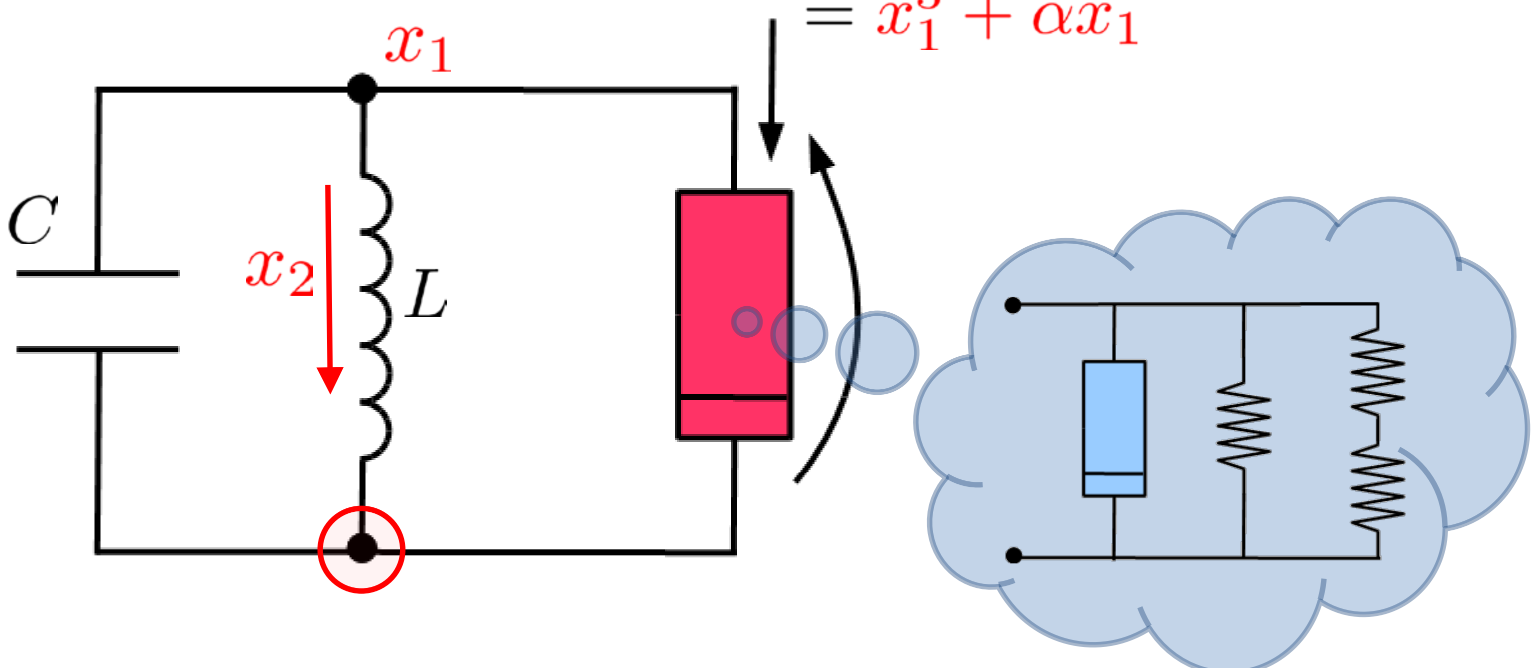


$$
\begin{cases}
C\dot{x}_1 + x_2 + x_1^3 + \dfrac{x_1}{R_1} + \dfrac{x_1 - x_3}{R_2} = 0 & \text{KCL} \\
L\dot{x}_2 - x_1 = 0 & \text{KVL} \\
\dfrac{x_1 - x_3}{R_2} - \dfrac{x_3}{R_3} = 0 & \text{KCL}
\end{cases}
$$

Modified nodal analysis (MNA)

The MNA model of non linear dynamical circuits is a set of Differential and Algebraic Equations constituting a DAE but … we will focus on the Ordinary Differential Equation (ODE) scenario only.

$$
\begin{aligned}
I_{NL} &= \\
&= V_{NL}^3 + \alpha V_{NL} = \\
&= x_1^3 + \alpha x_1
\end{aligned}
$$



$$
\begin{cases}
C\dot{x}_1 + x_2 + x_1^3 + \alpha x_1 = 0 \\
L\dot{x}_2 = x_1
\end{cases}
$$

Modified nodal analysis (MNA)

The MNA model of non linear dynamical circuits is a set of Differential and Algebraic Equations constituting a DAE but … we will focus on the Ordinary Differential Equation (ODE) scenario only.

We can do that without loss of generality since:

1. it would be much more complicated but possible to directly work on DAEs;
2. in general a transformation always exists that allows to reformulate a circuit in such a way that DAEs naturally becomes ODEs (we should adopt two-port ideal power transferitors and gyrators).

$$\begin{cases} \dfrac{d}{dt} f_{Q\Phi}(x(t)) + f_{VI}(x(t)) + B u_{VI}(t) = 0 \\ x \in \mathbb{R}^N \\ u_{VI} \in \mathbb{R}^P \qquad B \in \mathbb{R}^{N \times P} \\ f_{Q\Phi}, f_{VI} : \mathbb{R}^N \to \mathbb{R}^N \end{cases}$$

Modified nodal analysis (MNA)

$$\begin{cases} \dfrac{d}{dt} f_{Q\Phi}(x(t)) + f_{VI}(x(t)) + Bu_{VI}(t) = 0 \\ f_{Q\Phi}, f_{VI} : \mathbb{R}^N \to \mathbb{R}^N \end{cases}$$

In general we assume to deal with non linear links between charge and voltage and magnetic-flux and current.

$$\begin{cases} F_{Q\Phi} \dfrac{d}{dt} x(t) + f_{VI}(x(t)) + Bu_{VI}(t) = 0 \\ F_{Q\Phi} \in \mathbb{R}^{N \times N} \\ \det(F_{Q\Phi}) \neq 0 \end{cases}$$

In this case, as in our example, capacitors and inductors are linear.

$$F_{Q\Phi}^{-1} \left[F_{Q\Phi} \frac{d}{dt} x(t) + f_{VI}(x(t)) + Bu_{VI}(t) \right] = 0$$

$$\frac{d}{dt} x(t) = f(x(t), t)$$

Summary

something about what one should know when dealing with circuits (and systems) described by a smooth vector field working at periodic steady state

- Modified Nodal Analysis Formulation (this is for circuits)
- **Steady State Analysis (time-domain shooting method)**
- Periodic small signal analysis
- Floquet Theory (stability)

Steady State Analysis (time-domain shooting method)

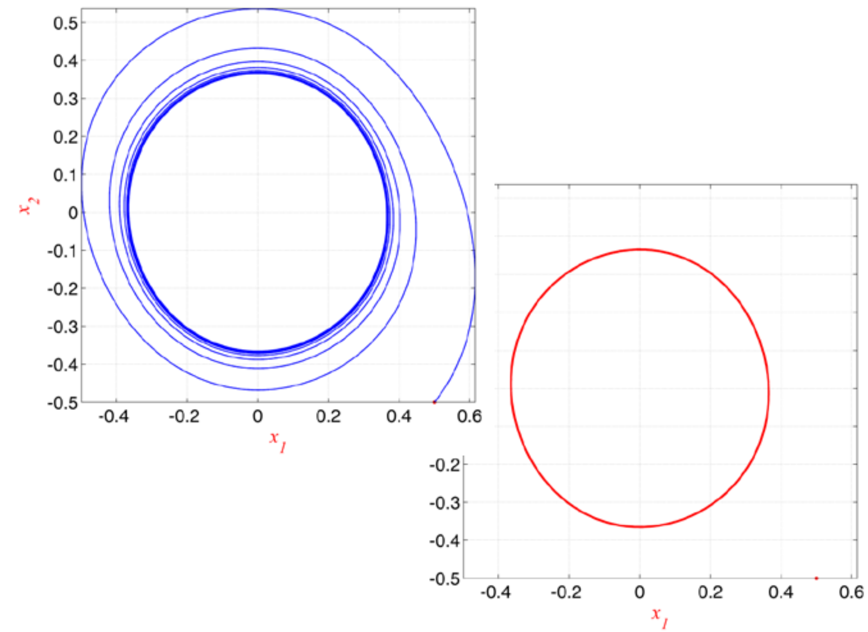
Smooth Circuit/System

$$\begin{cases} \dot{x} = f(x,t) \\ x(t_0) = x_0 \\ x(t) \in U \subset \mathbb{R}^N \\ f : \mathbb{R}^{N+1} \to \mathbb{R}^N \\ f \in C^1(\mathbb{R}^{N+1}) \end{cases}$$

Goal: efficiently find a **periodic steady state** solution of the ODE, i.e., a limit cycle (say γ)

$$\begin{cases} x_s(t) = x_s(t+T) \\ x_s(t_0) = \hat{x}_0 \in \gamma \end{cases}$$

Efficiently means that we do not want to perform a long lasting transient analysis to obtain the steady state behavior but we aim at directly find it.



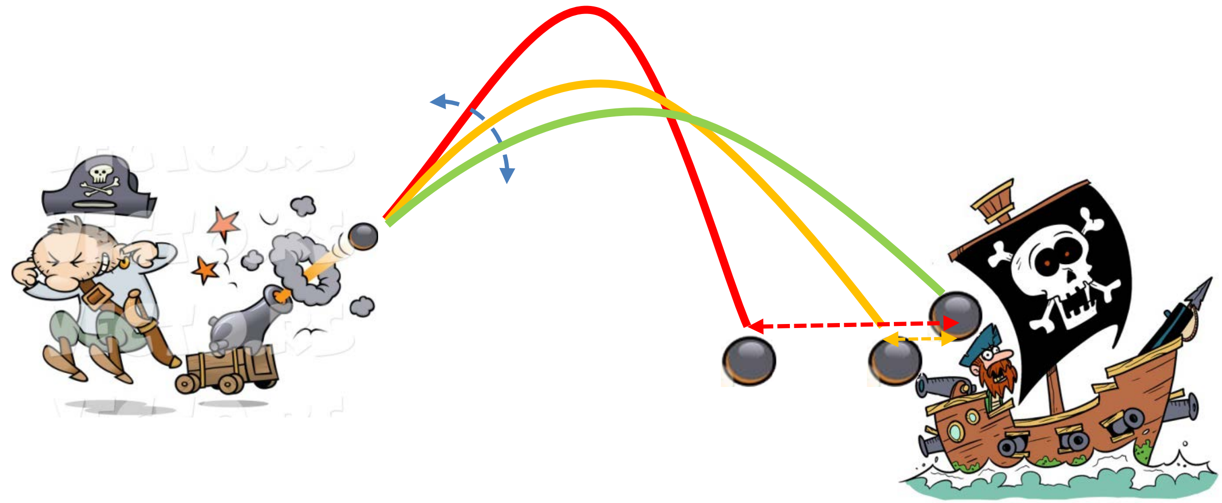
This is not a whim!

For instance, if we design a high-Q oscillator that long lasting transient analysis could take hours!

Steady State Analysis (time-domain shooting method)

Smooth Circuit/System

$$\begin{cases} \dot{x} = f(x,t) \\ x(t_0) = x_0 \\ x(t) \in U \subset \mathbb{R}^N \\ f : \mathbb{R}^{N+1} \to \mathbb{R}^N \\ f \in C^1(\mathbb{R}^{N+1}) \end{cases}$$



Goal: efficiently find a **periodic steady state** solution of the ODE, i.e., a limit cycle (say γ)

$$\begin{cases} x_s(t) = x_s(t+T) \\ x_s(t_0) = \hat{x}_0 \in \gamma \end{cases}$$

If a first shot misses the target, the gunner will change the tilt of the cannon, evaluate how much closer or farther he gets from his objective and finally adjust the tilt in order to (hopefully) hit the target with the next shot.

The key of the gunner's method is the perturbation of the initial guess and evaluation of the *sensitivity* of the solution (the arrival position of the cannonball) to this perturbation.

Steady State Analysis (time-domain shooting method)

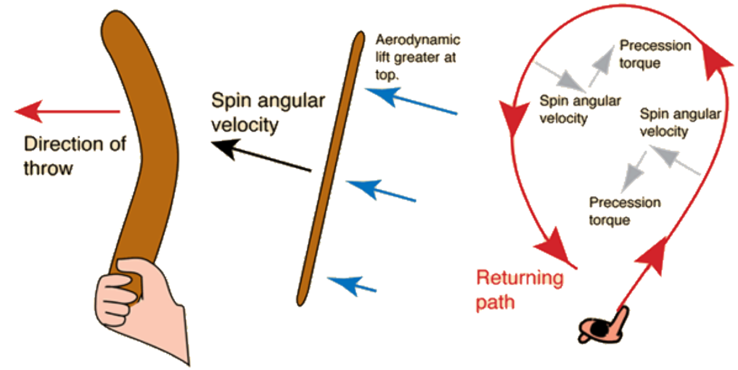
Smooth Circuit/System

$$
\begin{cases}
\dot{x} = f(x,t) \\
x(t_0) = x_0 \\
x(t) \in U \subset \mathbb{R}^N \\
f : \mathbb{R}^{N+1} \to \mathbb{R}^N \\
f \in C^1(\mathbb{R}^{N+1})
\end{cases}
$$

Goal: efficiently find a **periodic steady state** solution of the ODE, i.e., a limit cycle (say γ)

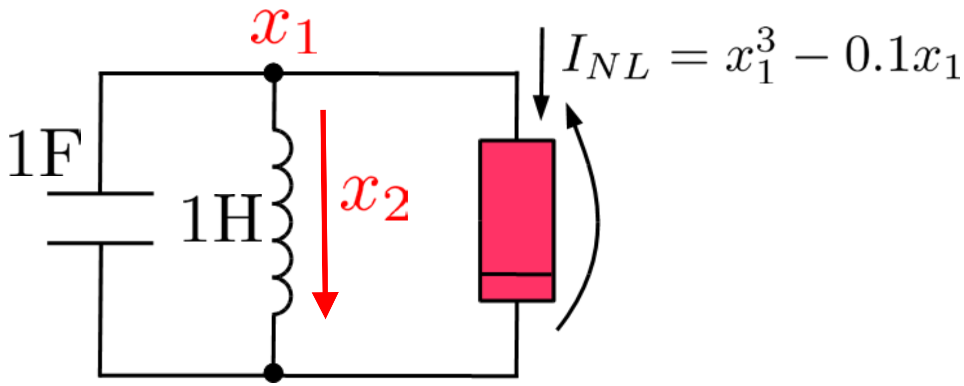
$$
\begin{cases}
x_s(t) = x_s(t+T) \\
x_s(t_0) = \hat{x}_0 \in \gamma
\end{cases}
$$

We are not gunners … we play with a boomerang since we are looking for a periodic trajectory (the initial point must coincide we the final one)

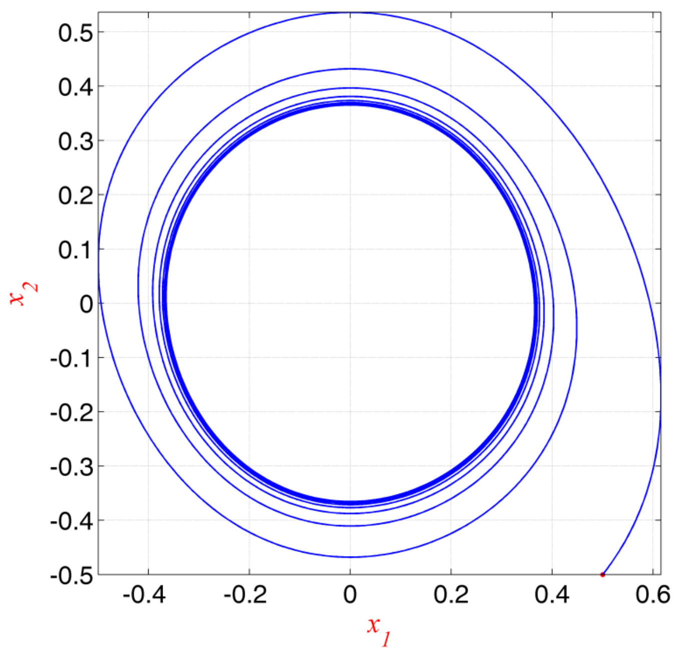


Do not worry! Shooting method is hopefully easier than throwing a boomerang … even if we do need some math!

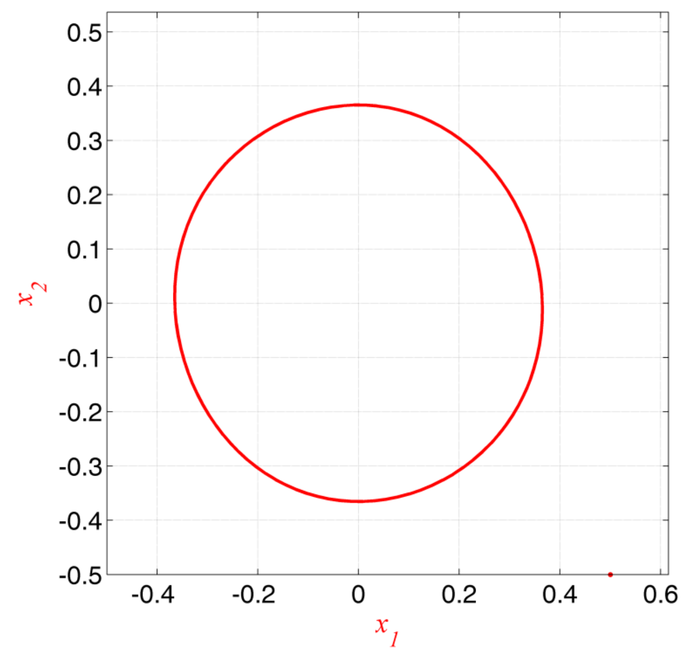
In our basic example … (Van der Pol oscillator)



$$\begin{cases} \dot{x}_1 = x_2 - (x_1^3 - 0.1x_1) \\ \dot{x}_2 = x_1 \\ x_1(0) = 0.5 \\ x_2(0) = -0.5 \end{cases}$$



Transient analysis



Shooting method (T= 6.2871s)

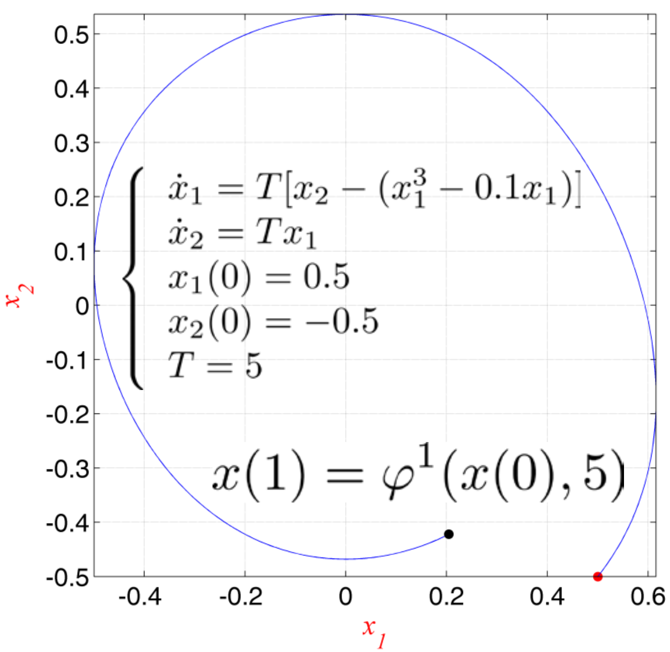
Steady State Analysis (time-domain shooting method)

We can formulate the problem "to locate the periodic steady state" as a Boundary Value Problem (BVP)

Let's assume that the circuit is autonomous (it does not explicitly depend on the t independent variable) and normalize the t time w.r.t. the unknown T period ($\tau = t/T$)
So doing the period becomes 1.

$$\begin{cases} \frac{d}{d\tau}x(\tau) = T f(x(\tau)) \\ x(0) = x_0 \end{cases}$$

$x(\tau) = \varphi^{\tau}(x(0), T)$ i.e., we introduce the state transition function from 0 to τ.

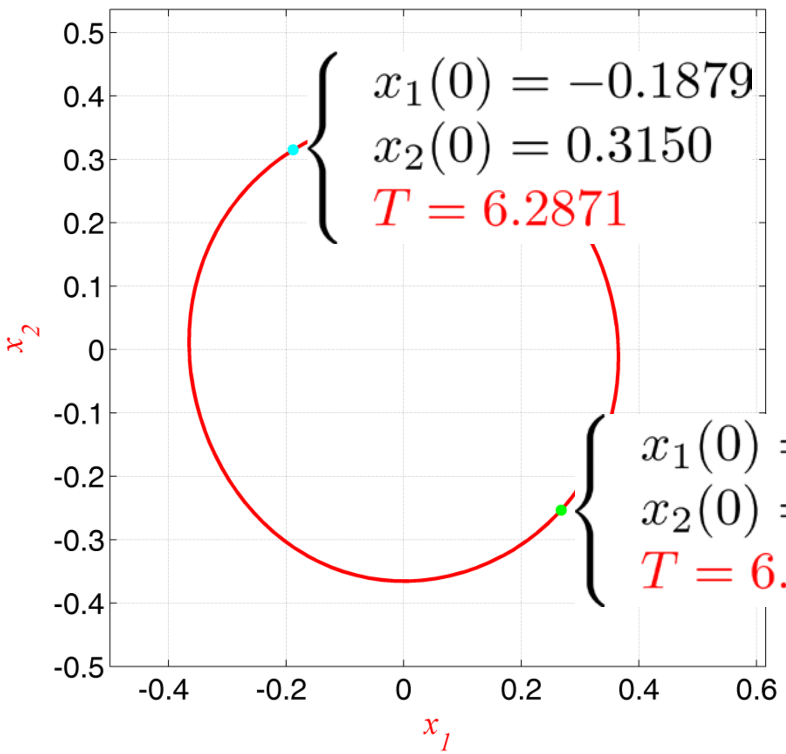


$$r(x(0), T) = \varphi^1(x(0), T) - x(0) \neq \mathbb{O}_N$$

Am I able to understand how I should modify the initial condition and the period that I guessed in such a way that the residue is null?

The sensitivity of the last point of the trajectory w.r.t. to both the initial one and the period should be available!

Steady State Analysis (time-domain shooting method)



$$\begin{cases} x_1(0) = -0.1879 \\ x_2(0) = 0.3150 \\ T = 6.2871 \end{cases}$$

$$\begin{cases} x_1(0) = 0.2677 \\ x_2(0) = -0.2533 \\ T = 6.2871 \end{cases}$$

To locate the initial condition and the period such that …

$$r(x(0),T) = \varphi^1(x(0),T) - x(0) = \mathbb{O}_N$$

… is a problem that admits infinite solution each one sliding on top of the others!

Let it be $s(x(0),T) = 0$ a proper "phase condition" that guarantees a unique solution of the BVP … it is a non linear algebraic equation depending on $(x(0),T)$

… the overall BVP becomes …

$$\begin{cases} r(x(0),T) = \varphi^1(x(0),T) - x(0) = \mathbb{O}_N \\ s(x(0),T) = 0 \end{cases}$$

Steady State Analysis (time-domain shooting method)

Things are easier if the circuit is non autonomous and, in particular, if it is periodically driven by voltage and/or a current sources.

In this case the phase condition is not necessary since the input signal gives itself a time reference and the T period is no longer an unknown.

We do not need to normalize the t time w.r.t. the T period.

$$\begin{cases} \dfrac{d}{d\tau}x(\tau) = T f(x(\tau)) \\ x(0) = x_0 \end{cases}$$

autonomous

(the initial time can be arbitrarily fixed to 0)

$$\begin{cases} \dfrac{d}{dt}x(t) = f(x(t), t) \\ x(t_0) = x_0 \end{cases}$$

non autonomous

I will present the theory in the autonomous case (the more general) but I will not show you how the phase condition can be chosen … if you are interested in it I can provide you with some references.

$$\begin{pmatrix} r(x(0),T) \\ s(x(0),T) \end{pmatrix} = \mathbb{O}_{N+1}$$

This BVP depends on N+1 unknowns and it is a set of N+1 non linear algebraic equations that can be numerically solved for instance using the iterative Newton's method.

$$\begin{pmatrix} r(x(0),T) \\ s(x(0),T) \end{pmatrix} = F(\underbrace{x(0),T}_{y}) = \mathbb{O}_{N+1}$$

$$\begin{cases} F(y) = \mathbb{O}_{\mathbb{R}^{N+1}} \\ F(y) : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1} \\ F \in C^1(\mathbb{R}^{N+1}) \\ \{J_F\}_{jk} = \frac{\partial}{\partial y_k} F_j(y) \end{cases}$$

Newton's method

At the *p*-th iteration of the method one has

$$y^p = (y_1^p, \dots, y_{N+1}^p)$$

which is an approximation of the solution

$$y^* = (y_1^*, \dots, y_{N+1}^*)$$

We assume that

$$y^* = y^p + \epsilon^p = (y_1^p + \epsilon_1^p, \dots, y_{N+1}^p + \epsilon_{N+1}^p)$$

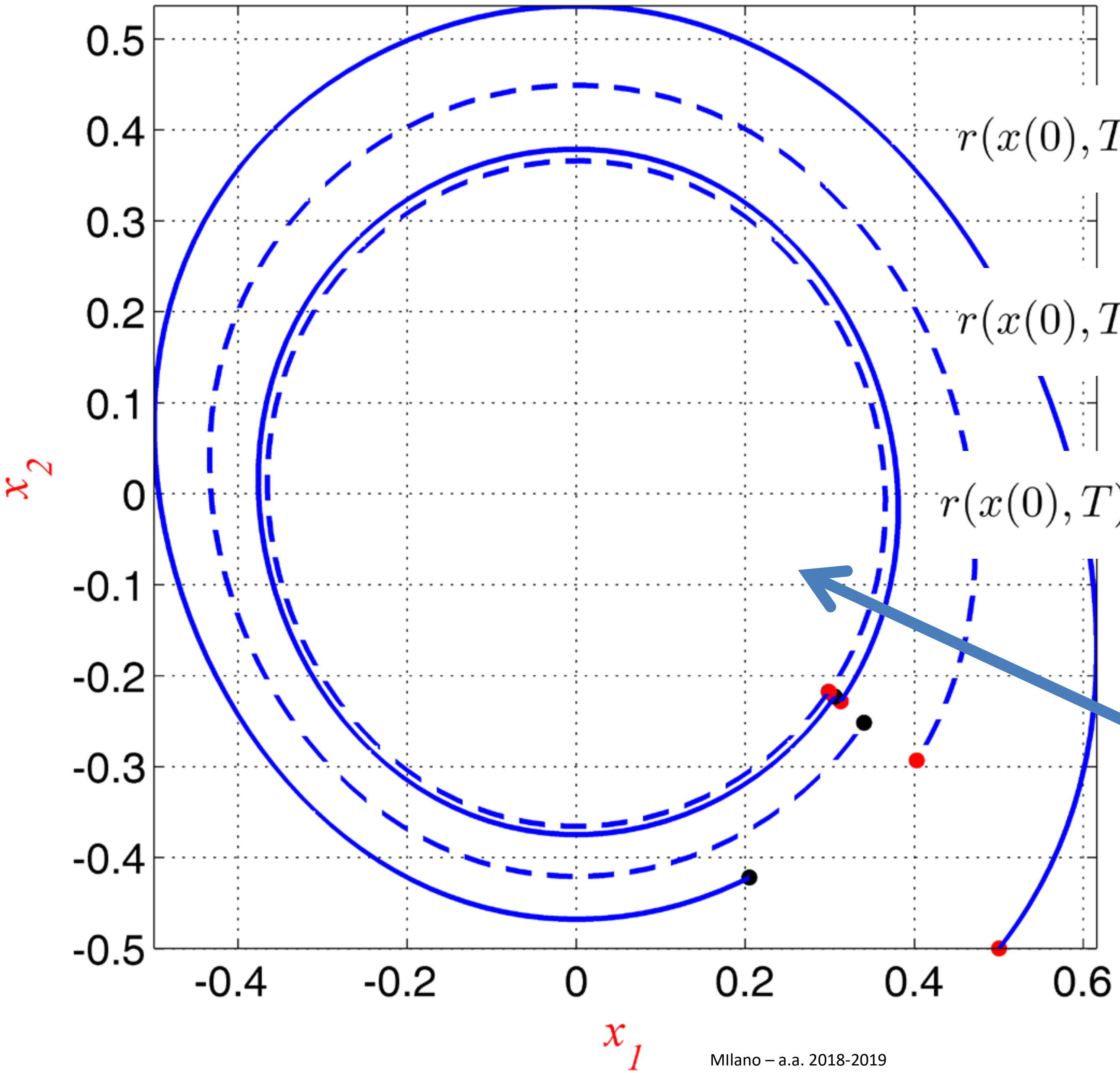
hence

$$F(y_1^p + \epsilon_1^p, \dots, y_{N+1}^p + \epsilon_{N+1}^p) = \mathbb{O}_{N+1}$$

$$F(y^p + \epsilon^p) \approx F(y^p) + J_F|_{y=y^p}\, \epsilon^p$$

$$\epsilon^p = -\left(J_F|_{y=y^p} \right)^{-1} F(y^p)$$

$$y^{p+1} = y^p + \epsilon^p$$



At fourth iteration:

$$r(x(0),T) = \begin{pmatrix} 2.3869 \cdot 10^{-4} \\ -1.7674 \cdot 10^{-4} \end{pmatrix}$$

At tenth iteration:

$$r(x(0),T) = \begin{pmatrix} 3.3051 \cdot 10^{-8} \\ -2.4479 \cdot 10^{-8} \end{pmatrix}$$

At eleventh iteration:

$$r(x(0),T) = \begin{pmatrix} -2.1450 \cdot 10^{-10} \\ 1.5886 \cdot 10^{-10} \end{pmatrix}$$

Steady State Analysis (time-domain shooting method)

$$\begin{cases} r(x(0),T) = \varphi^1(x(0),T) - x(0) = \mathbb{O}_N \\ s(x(0),T) = 0 \end{cases}$$

$$x^{(k+1)}(0) = x^{(k)}(0) + \Delta x^{(k)}(0)$$
$$T^{(k+1)} = T^{(k)} + \Delta T^{(k)}$$
$$k = k+1$$

$$\left(x^{(k)}(0), T^{(k)}\right)$$

$$\left\| \begin{pmatrix} r(x^{(k)}(0), T^{(k)}) \\ s(x^{(k)}(0), T^{(k)}) \end{pmatrix} \right\| < \epsilon_{rel} \left\| \begin{pmatrix} x^{(k)}(0) \\ T^{(k)} \end{pmatrix} \right\| + \epsilon_{abs}$$

Are you satisfied with current solution?

YES

END

NO

Trajectory sensitivity w.r.t initial conditions

$$\begin{pmatrix} \Delta x^{(k)}(0) \\ \Delta T^{(k)} \end{pmatrix} = -\left. \begin{pmatrix} \frac{\partial \varphi^\tau(x(0),T)}{\partial x(0)} - \mathbb{1}_N & \frac{\partial \varphi^\tau(x(0),T)}{\partial T} \\ \frac{\partial s(x(0),T)}{\partial x(0)} & \frac{\partial s(x(0),T)}{\partial T} \end{pmatrix} \right|_{\tau=1, x^{(k)}(0), T^{(k)}} \begin{pmatrix} r(x^{(k)}(0), T^{(k)}) \\ s(x^{(k)}(0), T^{(k)}) \end{pmatrix}$$

The original idea by Aprille and Trick (1972)

$$\begin{cases} r(x(0),T) = \varphi^1(x(0),T) - x(0) = \mathbb{O}_N \\ \cancel{s(x(0),T) = 0} \end{cases}$$

Let's assume to know the r-th entry of x(0) at the p-th iteration of the Newton method and define:

$$y^{(k)} = (x_1^{(k)}(0), ..., x_{r-1}^{(k)}(0), T^{(k)}, x_{r+1}^{(k)}(0), ..., x_N^{(k)}(0))$$

$$\widetilde{r}(y^{(k)}) = \varphi^1(y^{(k)}) - x(0) = \mathbb{O}_N$$

$$\Delta y^{(k)}(0) = -\left[\ \Lambda_{:,1\ldots r-1} \quad f(x^{(k)}(T^{(k)})) \quad \Lambda_{:,r+1\ldots N}\ \right] \widetilde{r}(y^{(k)})$$

$$\Lambda = \left(\frac{\partial \varphi^\tau(x(0),T)}{\partial x(0)} - \mathbb{1}_N \right) \Bigg|_{\tau=1, x^{(k)}(0), T^{(k)}}$$

$$y^{(k+1)} = \Delta y^{(k)} + y^{(k)} \qquad\qquad x_r^{(k+1)}(0) = x_r^{(k)}(T^{(k)})$$

Steady State Analysis (time-domain shooting method)

$$\begin{pmatrix} \Delta x^{(k)}(0) \\ \Delta T^{(k)} \end{pmatrix} = -\left.\begin{pmatrix} \frac{\partial \varphi^{\tau}(x(0),T)}{\partial x(0)} - \mathbb{1}_N & \frac{\partial \varphi^{\tau}(x(0),T)}{\partial T} \\ \frac{\partial s(x(0),T)}{\partial x(0)} & \frac{\partial s(x(0),T)}{\partial T} \end{pmatrix}\right|_{\tau=1,x^{(k)}(0),T^{(k)}} \begin{pmatrix} r(x^{(k)}(0),T^{(k)}) \\ s(x^{(k)}(0),T^{(k)}) \end{pmatrix}$$

$$\begin{cases} \frac{dx(\tau)}{d\tau} = Tf(x(\tau)) \\ x(0) = x_0 \end{cases}$$

$$\frac{\partial \varphi^{\tau}(x(0),T)}{\partial x(0)} = \frac{\partial x(\tau)}{\partial x(0)}$$

$$\begin{cases} \frac{\partial}{\partial x_0}\frac{d}{d\tau}x = \frac{\partial}{\partial x_0}Tf(x(\tau)) \\ \frac{\partial}{\partial x_0}x(0) = \frac{\partial}{\partial x_0}x_0 \end{cases}$$

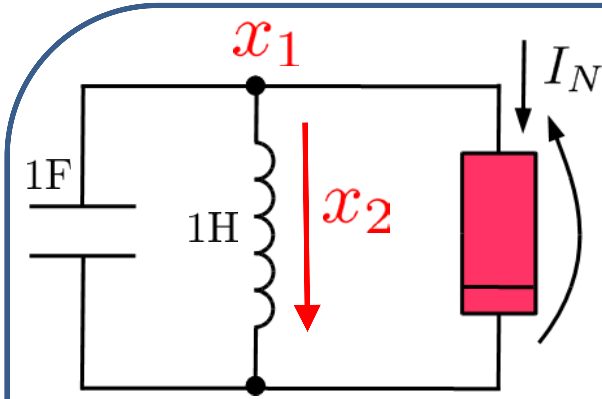
$$\begin{cases} \frac{d}{d\tau}\frac{\partial x}{\partial x_0} = T\frac{\partial f(x(\tau))}{\partial x}\frac{\partial x}{\partial x_0} \\ \frac{\partial x}{\partial x_0}(0) = \mathbb{1}_N \end{cases}$$

$$\begin{cases} \frac{\partial \dot{x}}{\partial x_0} = TJ_f(x(\tau))\frac{\partial x}{\partial x_0} \\ \frac{\partial x}{\partial x_0}(0) = \mathbb{1}_N \end{cases}$$

Linearized problem
Linear and time-varying ODE

$$\begin{cases} \dfrac{\partial \dot{x}}{\partial x_0} = T J_f(x(\tau)) \dfrac{\partial x}{\partial x_0} \\ \dfrac{\partial x}{\partial x_0}(0) = \mathbb{1}_N \end{cases}$$

Be careful … this is a LTV ODE that we want to solve N times with N linearly independent initial conditions! (the N columns of the identity matrix)



$$I_{NL} = x_1^3 - 0.1 x_1$$

$$\begin{cases} \dot{x}_1 = x_2 - (x_1^3 - 0.1 x_1) \\ \dot{x}_2 = x_1 \end{cases}$$

$$J_f = \begin{bmatrix} -(3x_1^2 - 0.1) & 1 \\ 1 & 0 \end{bmatrix} \qquad \frac{\partial x}{\partial x_0} \triangleq \Xi$$

$$\begin{cases} \begin{bmatrix} \dot{\Xi}_{11} & \dot{\Xi}_{21} \\ \dot{\Xi}_{21} & \dot{\Xi}_{22} \end{bmatrix} = T \begin{bmatrix} -(3x_1^2 - 0.1) & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \Xi_{11} & \Xi_{21} \\ \Xi_{21} & \Xi_{22} \end{bmatrix} \\ \Xi(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{cases}$$

Linear Time-varying Differential Equations

$$\begin{cases} \dot{\xi} = B(t)\xi + \beta(t) \\ \xi(t_0) = \xi_0 \in \mathbb{R}^N \end{cases}$$

If $B(t)$ and $\beta(t)$ are continuous in the $[t_0, t_1]$ interval then the problem admits one and only one solution.

$$\dot{\xi} = B(t)\xi$$

The set of all the solutions of the homogeneous ODE associated with the original problem is an N-dimensional subspace.

$$\phi(t) = \begin{bmatrix} \xi_1(t) & \cdots & \xi_N(t) \end{bmatrix}$$

$$\det(\phi(t) \neq 0) \ \forall t \geq t_0$$

A fundamental matrix contains by definition a basis of that subspace. The determinant of that matrix, must be different from zero.

$$C \quad | \quad det(C) \neq 0$$

$$\phi(t)C$$

It is a fundamental matrix too.

Linear Time-varying Differential Equations

$$\begin{cases} \dot{\xi} = B(t)\xi \\ \xi(t_0) = \xi_0 \end{cases}$$

It's unique solution is given by $\xi_s(t) = \phi(t)\phi^{-1}(t_0)\xi_0$

$\Phi(t, t_0) = \phi(t)\phi^{-1}(t_0)$ $\quad \Phi(t_0, t_0) = \mathbb{1}_N$ It is the canonical fundamental matrix.

$$\begin{cases} \dot{\xi} = B(t)\xi + \beta(t) \\ \xi(t_0) = \xi_0 \end{cases} \Rightarrow \begin{cases} \dot{\Phi}(t, t_0) = B(t)\Phi(t, t_0) \\ \Phi(t_0, t_0) = \mathbb{1}_N \end{cases}$$

Lagrange's formula

$$\xi_s(t) = \overbrace{\Phi(t, t_0)\xi_0}^{\xi_s^{free}(t) \triangleq \text{free response}} \underbrace{+ \Phi(t, t_0) \int_{t_0}^{t} \Phi^{-1}(\tau, t_0)\beta(\tau)d\tau}_{\xi_s^{forced}(t) \triangleq \text{forced response}}$$

Steady State Analysis (time-domain shooting method)

Trajectory sensitivity w.r.t initial conditions is provided by
the fundamental matrix computed along the trajectory
(during each iteration of the Newton method)

$$
\begin{cases}
\dfrac{\partial \dot{x}}{\partial x_0} = T J_f(x(\tau)) \dfrac{\partial x}{\partial x_0} \\
\dfrac{\partial x}{\partial x_0}(0) = \mathbb{1}_N
\end{cases}
\qquad
\frac{\partial x}{\partial x_0} = \Phi(\tau, 0)
\quad \Rightarrow \quad
\begin{cases}
\dot{\Phi}(\tau, 0) = T J_f(x(\tau)) \Phi(\tau, 0) \\
\Phi(0, 0) = \mathbb{1}_N
\end{cases}
$$

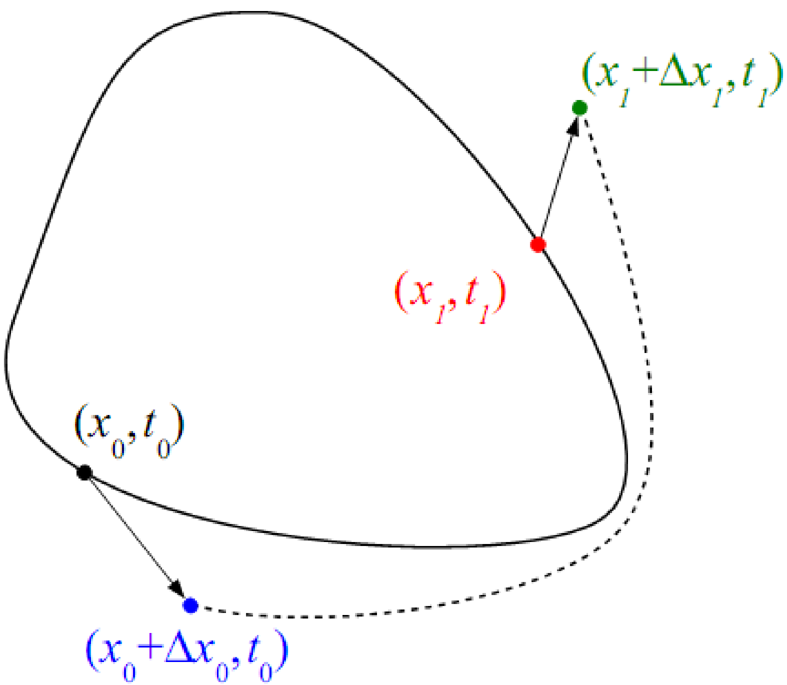
… once the periodic steady state solution has been found …

$$
\begin{cases}
\dfrac{dx(\tau)}{d\tau} = T f(x(\tau)) \\
x(0) = x_0 \\
\dot{\Phi}(\tau, 0) = T J_f(x(\tau)) \Phi(\tau, 0) \\
\Phi(0, 0) = \mathbb{1}_N
\end{cases}
\quad \Rightarrow \quad
\begin{cases}
x(0) \rightarrow x_s(0) \\
T \rightarrow T_s \\
x(t) \rightarrow x_s(t) = x_s(t + T_s) \\
J_f(x(t)) \rightarrow J_f(x_s(t)) = J_f^s(t) = J_f^s(t + T_s) \\
\Phi(t, 0) \rightarrow \Phi^s(t, 0) \\
\Phi^s(T_s, 0) \neq \Phi^s(0, 0)
\end{cases}
$$

(The solution of a PLTV it is not necessarily periodic)

The variational system

From a practical point of view the fundamental matrix can be obtained by performing a Forward Sensitivity Analysis (FSA) of the system trajectories with respect to the initial conditions or, in other words, solving the variational problem:



$$\begin{cases} \dot{x} = f(x,t)^{(*)} \\ x(t_0) = x_0 \\ \\ \dot{\Phi}(t,t_0) = J_f \Phi(t,t_0) \\ \Phi(t_0,t_0) = \mathbb{1}_N \end{cases}$$

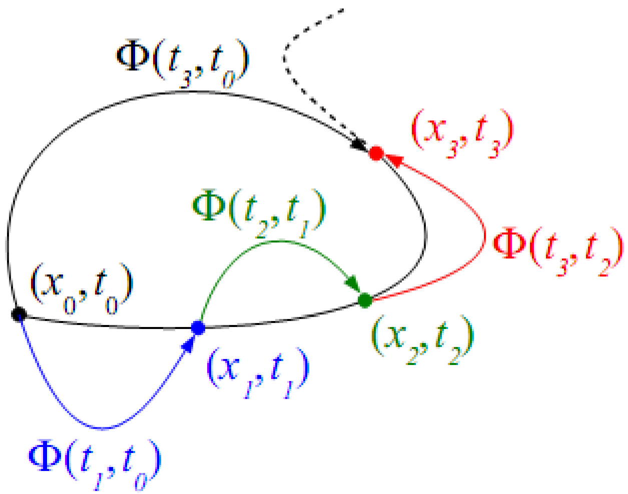
If $x_s(t)$ is the solution then it is possible to compute (at first order) the effect $\Delta x_s(t)$ on the solution produced by a small perturbation Δx_0 of the initial conditions

$$\Delta x_s(t) = \Phi(t,t_0)\ \Delta x_0(t_0)$$

(*) Note: I removed the multiplication by T since the t time is considered and not τ.

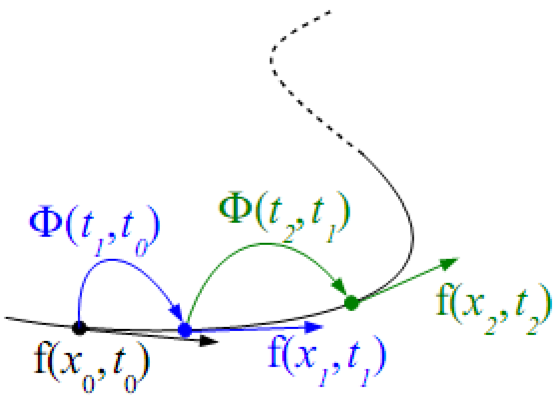
Some properties of the fundamental matrix

Composition property



$$\Phi(t_3,t_0) = \Phi(t_3,t_2)\Phi(t_2,t_1)\Phi(t_1,t_0)$$

Mapping property



$$f(x_1,t_1) = \Phi(t_1,t_0)f(x_0,t_0)$$
$$f(x_2,t_2) = \Phi(t_2,t_1)f(x_1,t_1)$$

Summary

something about what one should know when dealing with circuits (and systems) described by a smooth vector field and working at periodic steady state

- *Modified Nodal Analysis Formulation (this is for circuits)*
- *Periodic Steady State Analysis (time-domain shooting method)*
- ***Periodic small signal analysis***
- *Floquet Theory (stability)*

Analysis of the system in the neighborhood of the solution

Once the periodic steady state solution has been located, it is possible to resort to the linearization of the original system, in the neighborhood of the solution itself, to determine the effects of small periodic signals perturbing it (periodic small signal analysis ***PAC***).

$$u(t) = u(t + T_u)$$
$$mT_s = nT_u = \tilde{T}$$
$$n, m \in \mathbb{Z}$$

$$\begin{cases} \dot{x} = f(x) + u(t) \\ x(t_0) = x_0 \\ x(t) \in U \subset \mathbb{R}^N \\ f : \mathbb{R}^{N+1} \to \mathbb{R}^N \\ f \in C^1(\mathbb{R}^{N+1}) \end{cases}$$

Hp.

$$x_u(t) = x_s(t) + \zeta(t)$$
$$x_u(t_0) = x_0 + \zeta_0$$
$$x_u(t) = x_u(t + mT_s) = x_u(t + nT_u)$$

$$\frac{d}{dt}\left(x_s(t) + \zeta(t)\right) = f\left(x_s(t) + \zeta(t)\right) + u(t)$$

$$\dot{x}_s(t) + \dot{\zeta}(t) = f(x_s(t)) + J_f^s(t)\zeta(t) + O(||\zeta||^2) + u(t) \approx$$
$$\approx f(x_s(t)) + J_f^s(t)\zeta(t) + u(t)$$

$$\boxed{\begin{cases} \dot{\zeta}(t) = J_f^s(t)\zeta(t) + u(t) \\ \zeta(t_0) = \zeta_0 \end{cases}}$$

Analysis of the system in the neighborhood of the solution

$$\begin{cases} \dot{\zeta}(t) = J_f^s(t)\zeta(t) + u(t) \\ \zeta(t_0) = \zeta_0 \end{cases}$$

$$\zeta(t) = \Phi^s(t, t_0)\zeta_0 + \Phi^s(t, t_0) \int_{t_0}^{t} \left(\Phi^s(\tau, t_0)\right)^{-1} u(\tau) d\tau$$

It is available as a byproduct of the shooting method

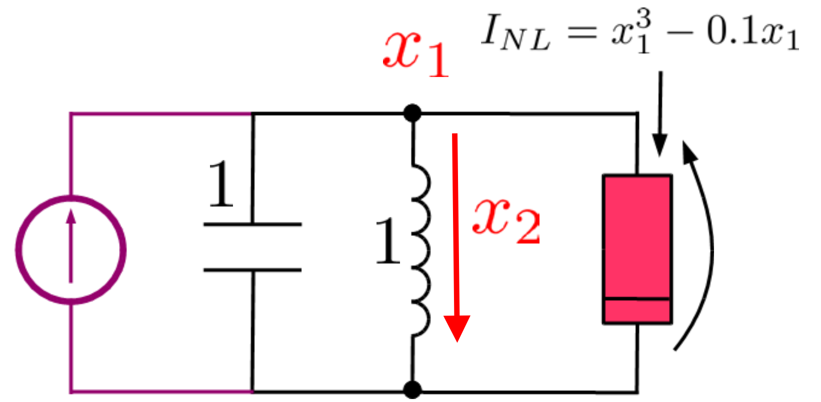
and

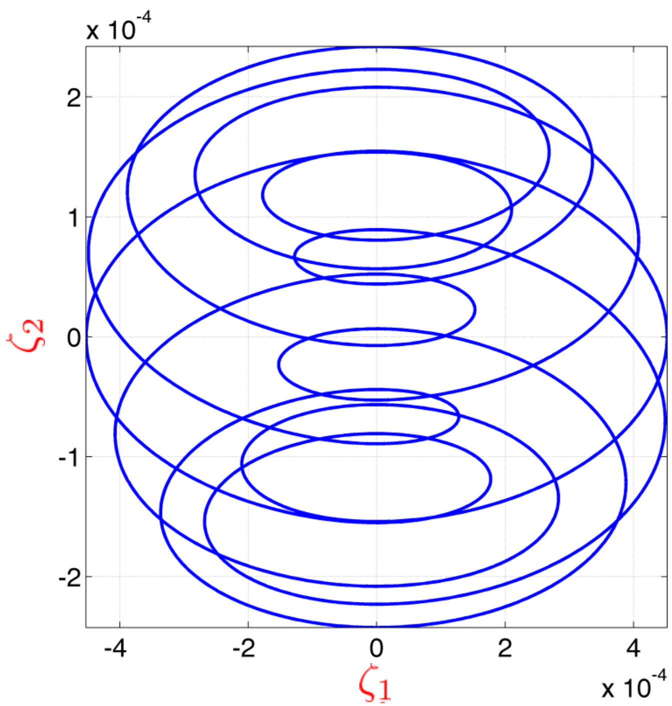
if you know it in $[t_0, t_0 + T_s]$ then, owing to the composition property, you know it for t≥0 (think about it!).

Analysis of the system in the neighborhood of the solution

$$\begin{cases} \dot{x}_1 = -x_2 - (x_1^3 - 0.1x_1) + u(t) \\ \dot{x}_2 = x_1 \\ x_1(0) = -0.1879 \\ x_2(0) = 0.3150 \end{cases}$$

$$u(t) = u(t + T_u)$$
$$\tilde{T} = 3T_s = 11T_u$$

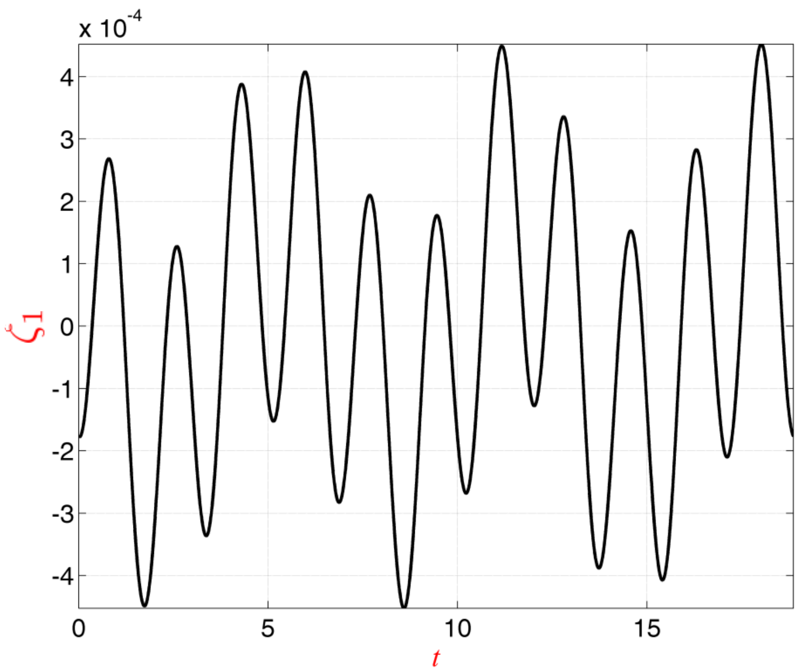




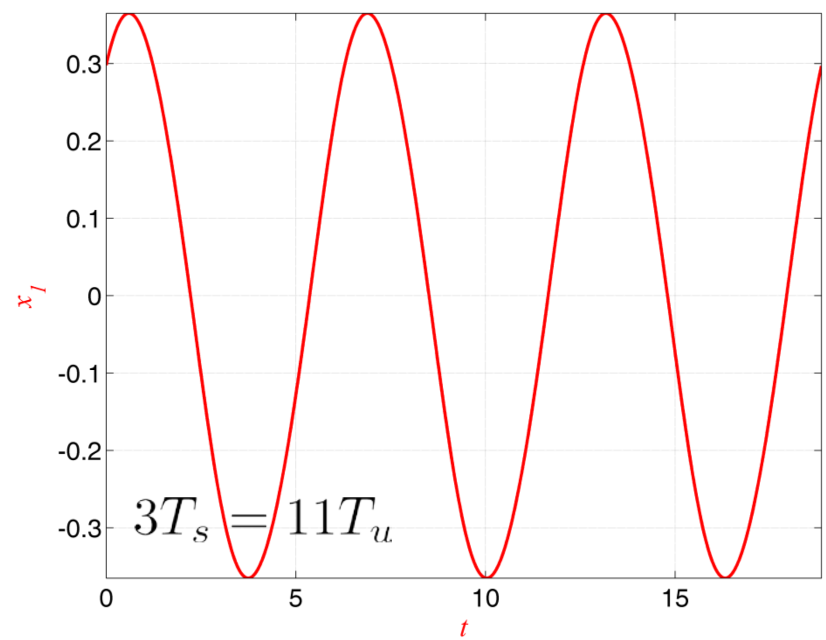
$$T_s = 6.2871$$

$$u(t) = 10^{-3} sin\left(2\pi \frac{t}{T_u}\right)$$

Analysis of the system in the neighborhood of the solution



"Small signal" trajectory



"Large signal" trajectory

Summary

something about what one should know when dealing with circuits (and systems) described by a smooth vector field and working at periodic steady state

- ***Modified Nodal Analysis Formulation (this is for circuits)***
- ***Periodic Steady State Analysis (time-domain shooting method)***
- ***Periodic small signal analysis***
- ***Floquet Theory (stability)***

Floquet Theory

$$\begin{cases} \dot{\xi} = B(t)\xi \\ B(t) = B(t + \mathrm{T}_s) \\ \xi(t_0) = \xi_0 \end{cases}$$

$$\begin{cases} \Phi(t, t_0) = \phi(t)\phi^{-1}(t_0) \\ \Phi(t_0, t_0) = \mathbb{1}_N \end{cases}$$

Floquet (1883)

$$\begin{cases} \Phi(t, t_0) = P(t, t_0)e^{F(t-t_0)} \\ P(t, t_0) = P(t + T_s, t_0) \\ P(t_0, t_0) = P(T_s + t_0, t_0) = \mathbb{1}_N \\ \det(P(t, t_0)) \neq 0 \quad \forall\, t, t_0 \end{cases}$$

Monodromy matrix

$$\begin{cases} \Phi(T_s + t_0, t_0) = \mathrm{e}^{FT_s} \\ F = \dfrac{1}{T_s}\log(\Phi(T_s + t_0, t_0)) \end{cases}$$

Let's assume that

$$\begin{cases} F = UDU^{-1} \\ D = \mathrm{diag}(\eta_i), \quad i = 1, \cdots, N \\ \{\eta_i\} = \mathrm{eig}(F) \end{cases}$$

Floquet exponents (Hp. N distinct)

$$\{\eta_i\} = \mathrm{eig}(F)$$

Floquet multipliers

$$\mathrm{eig}(\Phi(T_S + t_0, t_0)) = \mathrm{eig}(\mathbb{1}e^{FT_s}) =$$

$$= \mathrm{eig}(e^{(UDU^{-1})T_s}) =$$

$$= \mathrm{eig}(Ue^{DT_s}U^{-1}) =$$

$$= e^{\eta_i T_s} = \mu_i$$

Decomposition

$$\Phi(t + t_0, t_0) = P(t, t_0)Ue^{D(t-t_0)}U^{-1}$$

Floquet Theory

$$\Lambda(t+t_0,t_0) = P(t,t_0)Ue^{D(t-t_0)}$$ It's a fundamental matrix …

$$\begin{cases} P(t,t_0)U = [\lambda_1(t),\cdots,\lambda_N(t)] \\ \lambda_i(t) = \lambda_i(t+T_s), \quad i=1,\cdots,N \\ \Lambda(t+t_0,t_0) = \left[e^{(t-t_0)\eta_1}\lambda_1(t),\cdots,e^{(t-t_0)\eta_N}\lambda_N(t)\right] \end{cases}$$

The solution of the original problem can be expressed, by definition, as a linear combination of the columns of the fundamental matrix.

$$\xi_s(t) = c_1 e^{(t-t_0)\eta_1}\lambda_1(t) + \cdots + c_N e^{(t-t_0)\eta_N}\lambda_N(t)$$

The c_i constants depends on the initial conditions.

Periodicity: a null exponent ($e^{\eta_i T_s} = \mu_i \rightarrow$ at least a multiplier equal to 1)
Stability: all the other exponents must have negative real part (viz., multipliers in the unit circle in the complex plane)

Floquet Theory

Let's go back to our very problem …

$$
\begin{cases}
\dfrac{dx(\tau)}{d\tau} = T f(x(\tau)) \\
x(0) = x_0 \\
\dot{\Phi}(\tau,0) = T J_f(x(\tau))\Phi(\tau,0) \\
\Phi(0,0) = \mathbb{1}_N
\end{cases}
\qquad
\begin{cases}
x(0) \rightarrow x_s(0) \\
T \rightarrow T_s \\
x(t) \rightarrow x_s(t) = x_s(t+T_s) \\
J_f(x(t)) \rightarrow J_f(x_s(t)) = J_f^s(t) = J_f^s(t+T_s) \\
\Phi(t,0) \rightarrow \Phi^s(t,0)
\end{cases}
$$

Once a periodic solution is obtained, the variational equation becomes a LTV ODE with periodic coefficients and consequently each one of its columns can be written as

$$\Phi^s_{:,j}(t) = c_{1,j} e^{(t-t_0)\eta_1} \lambda_1(t) + \cdots + c_{1,N} e^{(t-t_0)\eta_N} \lambda_N(t)$$

The eigenvalues of the monodromy matrix are the Floquet multipliers and can be used to study the stability of the periodic steady state orbit.

As a matter of fact, if we apply a small perturbation at any point of that periodic orbit, the destiny of such perturbation depends on the Floquet multipliers.

Floquet Theory

The stable periodic steady state solution of an autonomous ODE always exhibits a multiplier equal to 1 whereas in the non autonomous case all the multipliers are within the unit circle in the complex plane.

Autonomous case

$$\begin{cases} \dot{x} = f(x,t) \\ x(t_0) = x_0 \\ x(t) \in U \subset \mathbb{R}^N \\ f : \mathbb{R}^{N+1} \to \mathbb{R}^N \\ f \in C^1(\mathbb{R}^{N+1}) \end{cases} \Rightarrow \begin{cases} x_s(t) = x_s(t+T) \\ x_s(t_0) = \hat{x}_0 \in \gamma \end{cases}$$

$$\frac{d}{dt}\dot{x}_s = \frac{d}{dt} f(x_s)$$

$$\Downarrow$$

$$\frac{d}{dt}\dot{x}_s = J_f(x_s)(\dot{x}_s)$$

$$\Downarrow$$

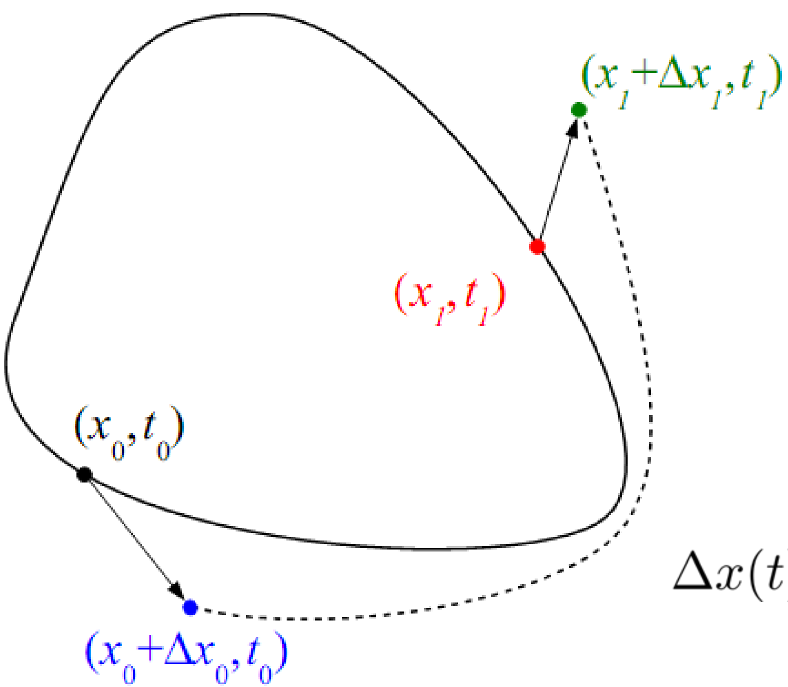
It is a solution of the linearized problem!

$$\dot{x}_s(t) = \sum_{i=1}^{n} c_i e^{(t-t_0)\eta_i} \lambda_i(t)$$

Since $\dot{x}_s(t)$ is periodic at least one exponent must be 0, viz., a Floquet multiplier is 1.

Non autonomous case: you can try!

Floquet Theory

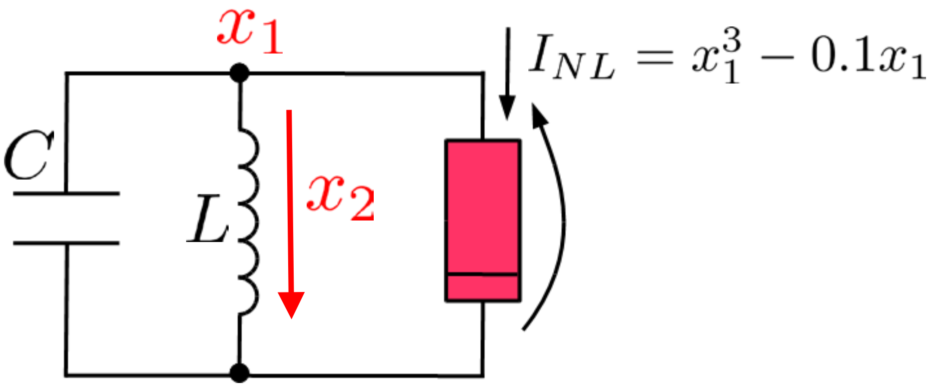


$$\Delta x(t) = \Phi^s(t, t_0)\Delta x_0$$

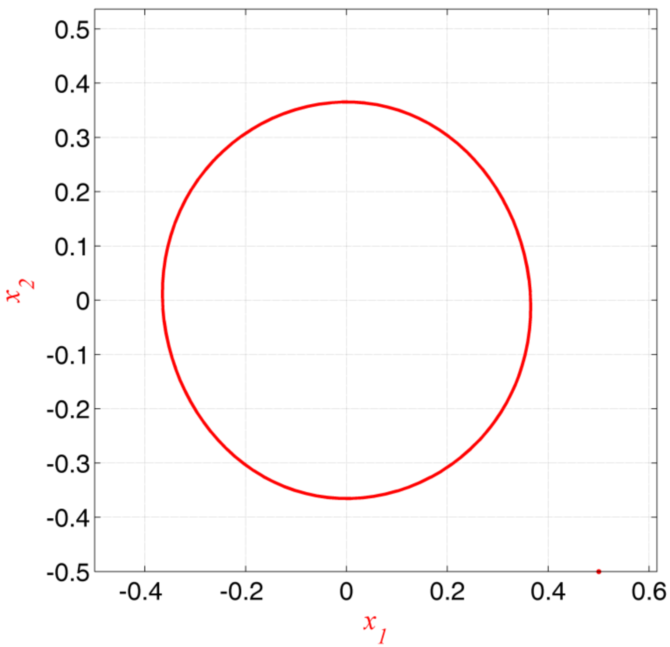
$$\begin{cases} \Phi^s(t,t_0) = P(t,t_0)e^{F(t-t_0)} \\ P(t,t_0) = P(t+T_s,t_0) \\ P(t_0,t_0) = P(T_s+t_0,t_0) = \mathbb{1}_N \\ \det(P(t,t_0)) \neq 0 \quad \forall t, t_0 \end{cases}$$

$$\Delta x(t) = c_1 e^{(t-t_0)\eta_1}\lambda_1(t) + \cdots + c_N e^{(t-t_0)\eta_N}\lambda_N(t)$$

If all the exponents have negative real part (viz., multipliers in the unit circle in the complex plane) the effect of the perturbation vanishes … if there is a multiplier equal to 1 the effect of the perturbation does not vanish!



$$\begin{cases} C\dot{x}_1 = -x_2 - (x_1^3 - 0.1x_1) \\ L\dot{x}_2 = x_1 \\ x_1(0) = 0.5 \\ x_2(0) = -0.5 \\ C = 1 \\ L = 1 \end{cases}$$



$$\Phi(T + t_0, t_0) = \begin{bmatrix} 0.69171 & 0.22836 \\ 0.21417 & 0.84136 \end{bmatrix}$$

$$eig(\Phi(T + t_0, t_0)) = \begin{bmatrix} 0.533069259221368 9 \\ 0.999999999861733 \end{bmatrix}$$

Shooting (T= 6.2871s)

… summing up …

The fundamental matrix is crucial for

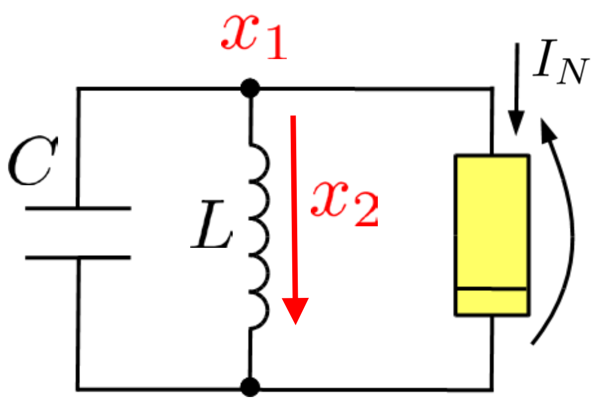
- Shooting method
- Small signal analysis
- Stability analysis

Variational system

$$\begin{cases} \dot{x} = f(x,t) \\ x(t_0) = x_0 \\ \dot{\Phi}(t,t_0) = J_f(x(t))\Phi(t,t_0) \\ \Phi(t_0,t_0) = \mathbb{1}_N \end{cases}$$

The Jacobian matrix of the *f* vector field must exist for every *x* and for every *t*.

What happens if …



$$I_{NL} = x_1^3 - 0.1x_1 + 0.1sign(x_1)$$

$$\begin{cases} C\dot{x}_1 = -x_2 - (x_1^3 - 0.1x_1 + 0.1sign(x_1)) \\ L\dot{x}_2 = x_1 \\ x_1(0) = 0.5 \\ x_2(0) = -0.5 \\ C = 1 \\ L = 1 \end{cases}$$

The constitutive equation of the non linear resistor becomes piecewise smooth and the dynamics of the circuit can be described by resorting to two vector fields according to the value of x_1:

$$\begin{cases} C\dot{x}_1 = -x_2 - (x_1^3 - 0.1x_1 + 0.1), & x_1 > 0 \\ L\dot{x}_2 = x_1 \end{cases}$$

$$\begin{cases} C\dot{x}_1 = -x_2 - (x_1^3 - 0.1x_1 - 0.1), & x_1 < 0 \\ L\dot{x}_2 = x_1 \end{cases}$$

The event x_1=0 rules the switching between the two vector fields and the 2D state space is partitioned by a surface

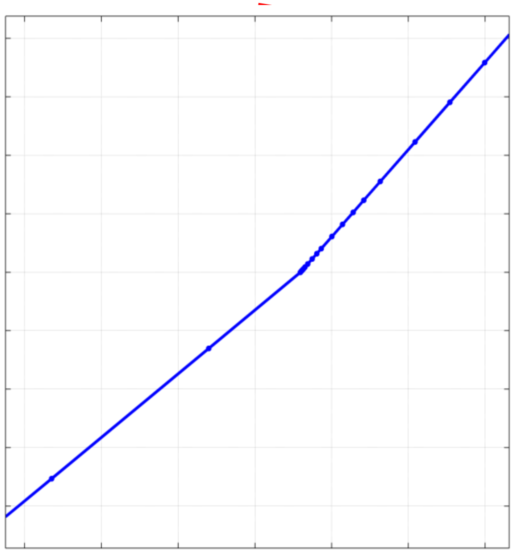
$$h(x_1, x_2) = x_1 = 0$$

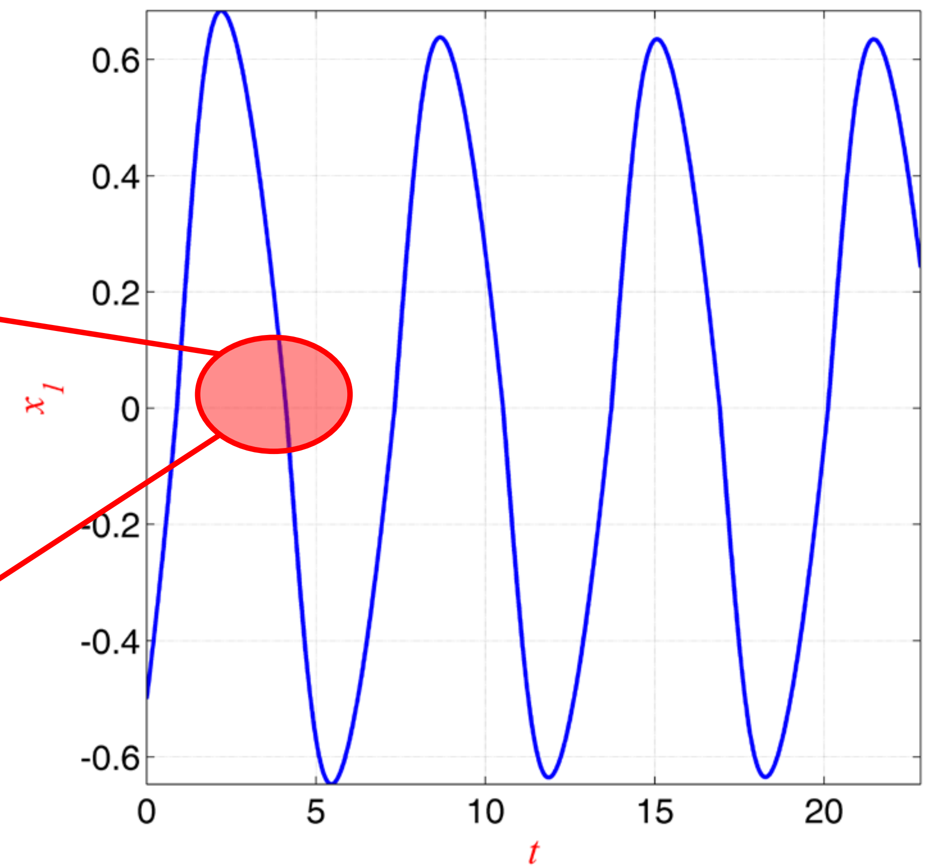
What happens if …

$$
\begin{cases}
C\dot{x}_1 = -x_2 - (x_1^3 - 0.1x_1 + 0.1), & x_1 > 0 \\
C\dot{x}_1 = -x_2 - (x_1^3 - 0.1x_1 - 0.1), & x_1 < 0 \\
L\dot{x}_2 = x_1 \\
x_1(0) = 0.5 \\
x_2(0) = -0.5 \\
C = 1 \\
L = 1
\end{cases}
$$

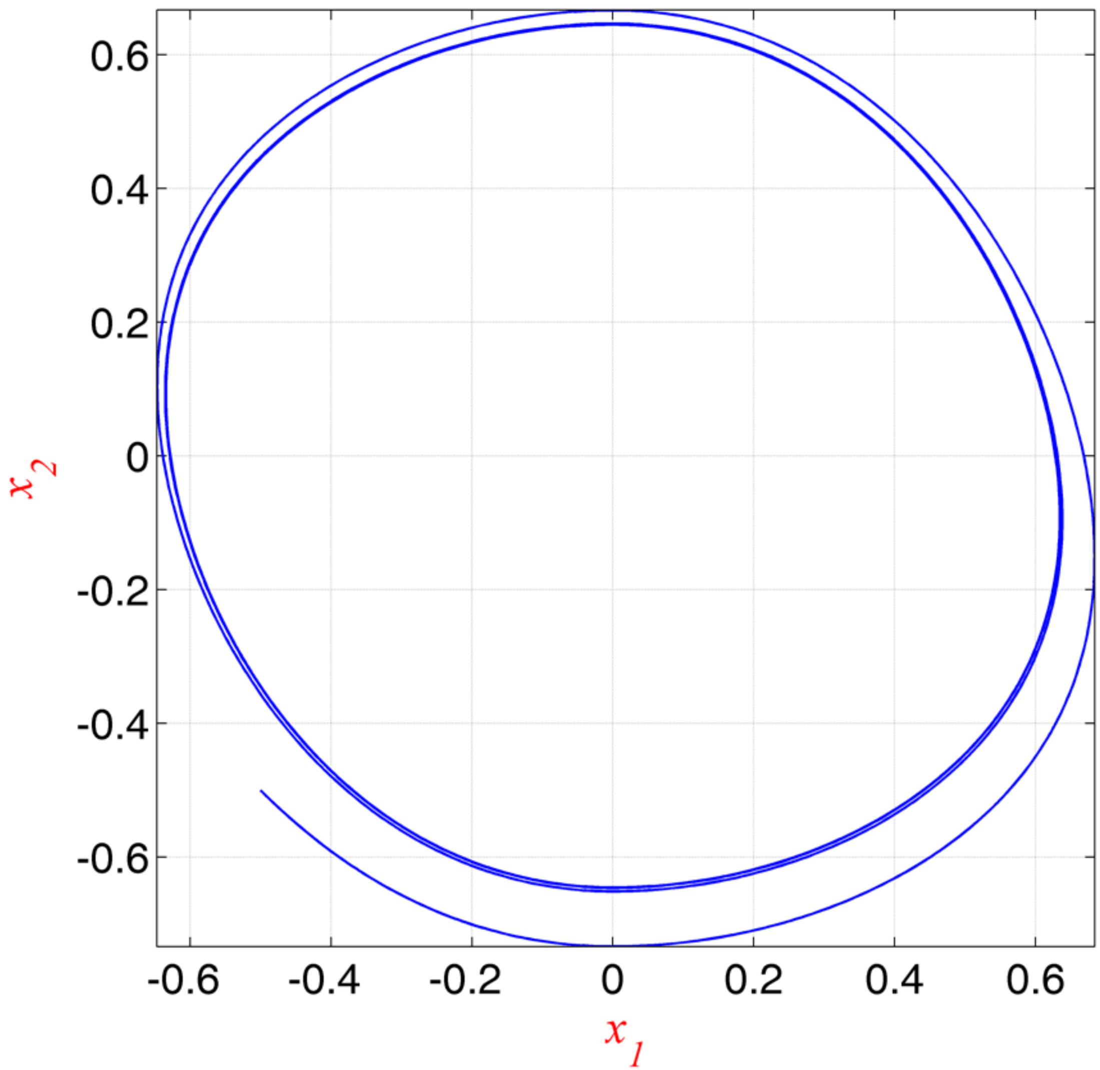
Let’s try to perform a transient simulation looking for the zero crossings of x_1

In correspondence with the
commutat
the sta
continuous
longer smo
tange
di





What happens if …



If we plot the result of our transient simulation and we plot the trajectory in the state space, I dare say that we should be convinced the a periodic steady state solution is admitted …

Could we use the shooting method and do everything as we did in the smooth case?

What happens if …

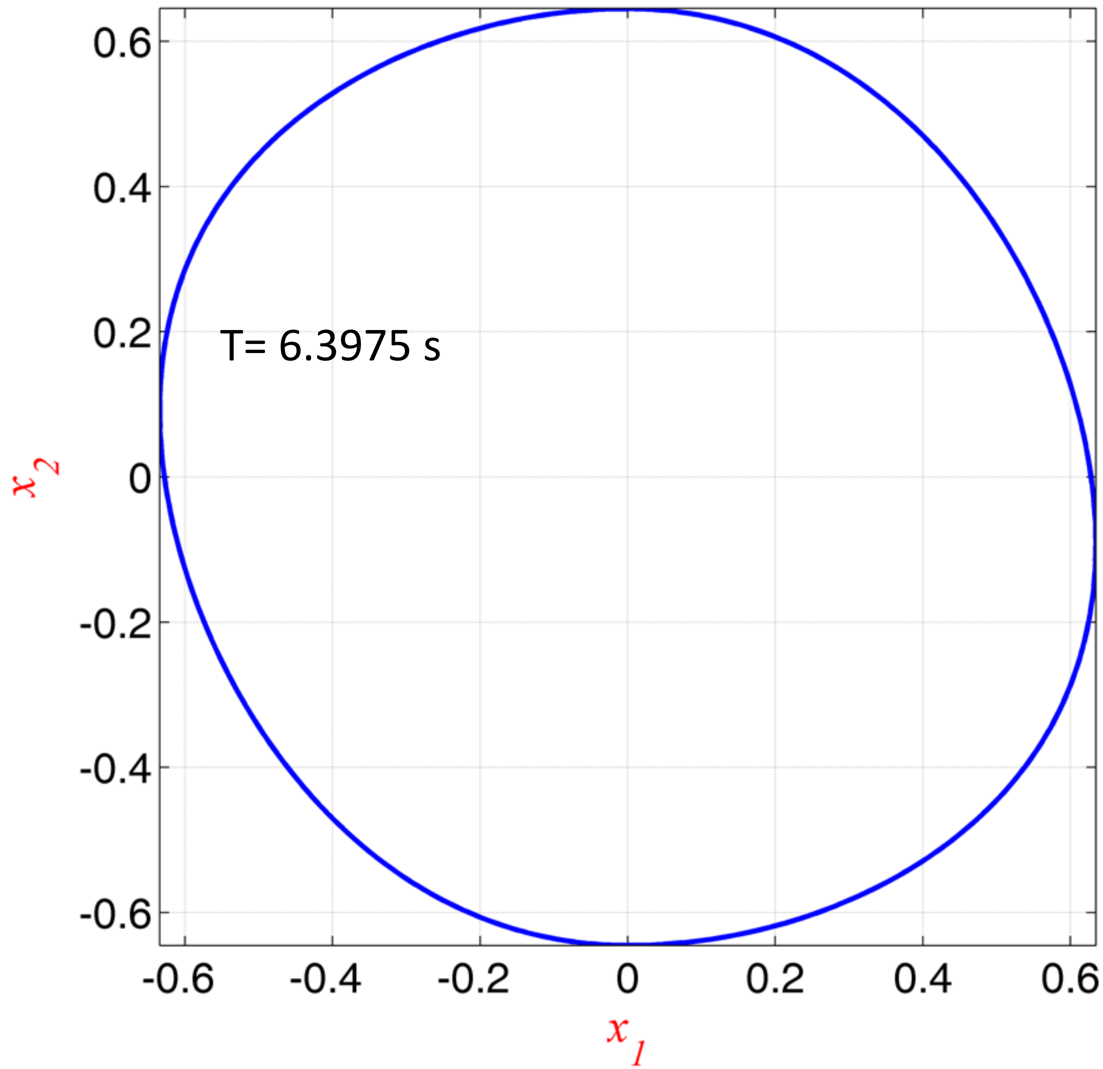
$$\begin{cases} \dot{x} = f(x,t) \\ x(t_0) = x_0 \\ \\ \dot{\Phi}(t,t_0) = J_f \Phi(t,t_0) \\ \Phi(t_0,t_0) = \mathbb{1}_N \end{cases}$$

$$\begin{cases} \dot{x}_1 = -x_2 - (x_1^3 - 0.1x_1 + 0.1), \quad x_1 > 0 \\ \dot{x}_1 = -x_2 - (x_1^3 - 0.1x_1 - 0.1), \quad x_1 < 0 \\ \dot{x}_2 = x_1 \\ \\ \dot{\Phi}(t,t_0) = J_f \Phi(t,t_0) = \begin{bmatrix} -3x_1^2 - 0.1 & -0.1 \\ 1 & 0 \end{bmatrix} \Phi(t,t_0) \\ \Phi(t_0,t_0) = \mathbb{1}_N \end{cases}$$

It is not valid in x_1=0

Let's apply the shooting method as we did before …

… it converges …



$$\Phi(T+t_0,t_0)=$$

$$=\begin{bmatrix} 0.3197 & -0.1883 \\ -0.2800 & 0.2909 \end{bmatrix}$$

$$eig(\Phi(T+t_0,t_0))=$$

$$\begin{bmatrix} 0.535334455722719 \\ 0.075217502808425 60 \end{bmatrix}$$

… but the monodromy matrix is wrong!

What happens if …

The switching system with the Jacobian that is not defined across the discontinuity is such that the Newton method succeeds. In fact, if the Jacobian is not "too wrong" the Newton method is able to correct such "small" mistakes owing to its iterative nature and robustness. In this case the discontinuity is moderated and so we found the limit cycle.

But we got it just thanks to the numerical method … the properties of linearization are completely lost.

We must do something better if we want to retain all the properties we derived in the smooth case (and also guarantee the convergence if the Jacobian is really "wrong")

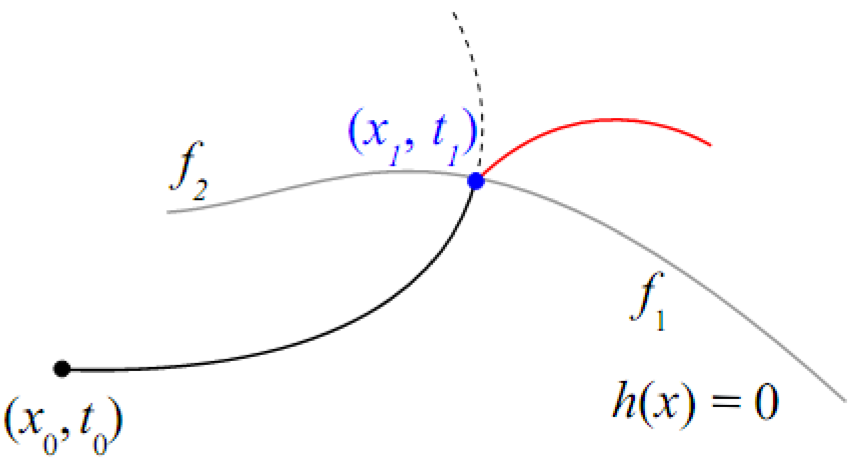
Hybrid dynamical systems

"Hybrid systems are made up of dynamical continuous/discrete time evolution processes interacting with logical/decisional processes"

Peters, K., & Parlitz, U. (2003). Hybrid systems forming strange billiards. International Journal of Bifurcation and Chaos in Applied Sciences and Engineering, 13(9), 2575– 2588.

"Switching" dynamical systems

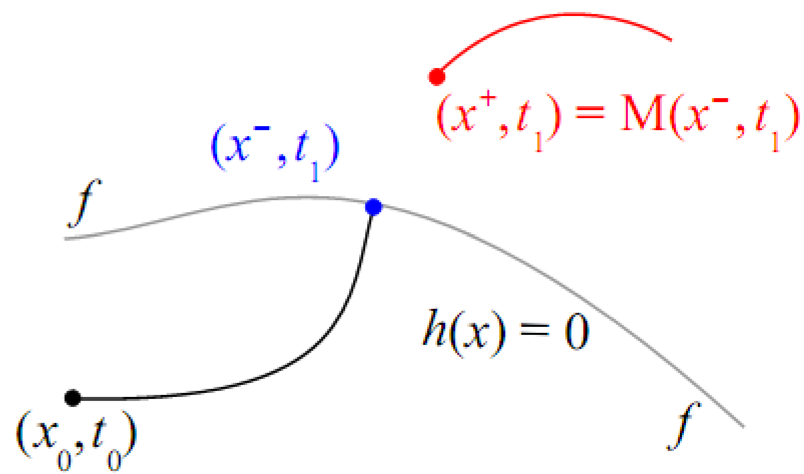
Discontinuous vector field but **no discontinuity in the state** (Filippov system)



(Continuous but not differentiable vector fields)

"Impact" systems

The **state** exhibits **discontinuity in time**



Hybrid dynamical systems

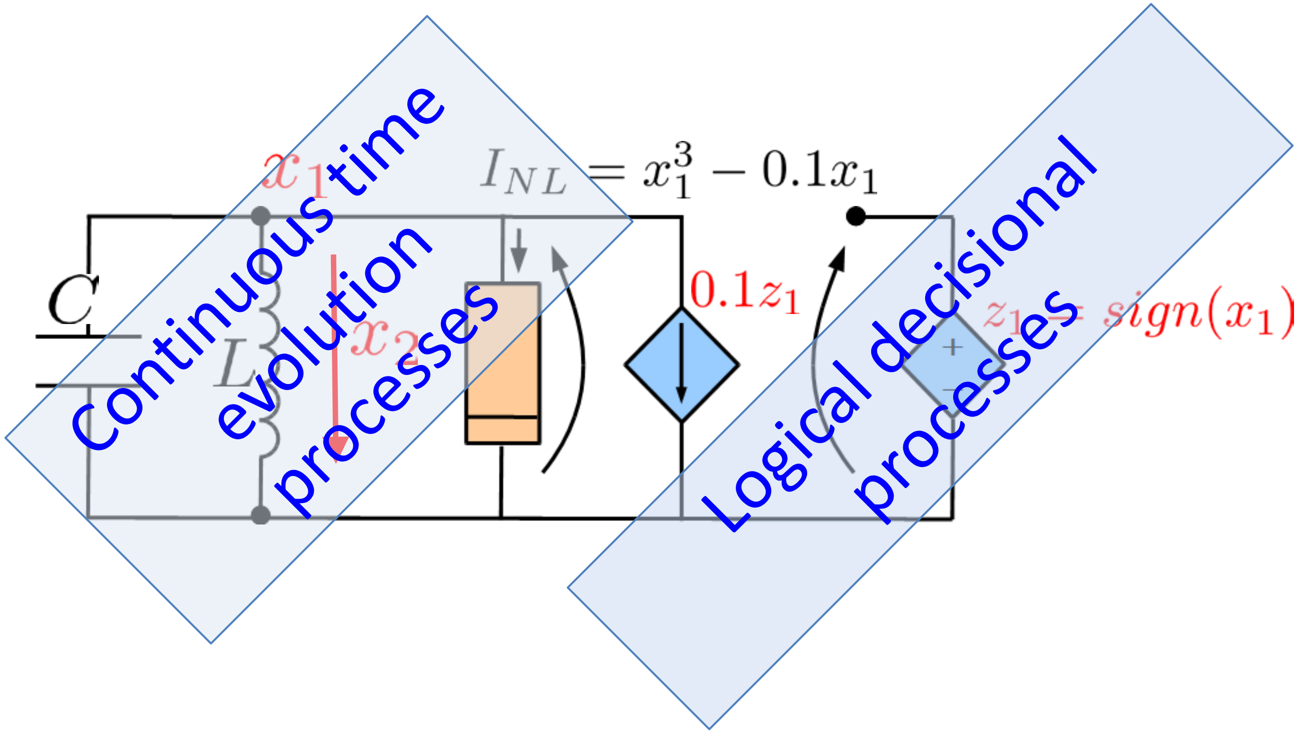
For hybrid dynamical systems the variational problem is not defined at **switching/impact points** since at those points the trajectory is not differentiable.

This problem can be overcome by resorting to a proper correction factor, the **saltation matrix**,* to be used at switching/impact points.

$$
\begin{cases}
\dot{x} = f(x,t) \\
x(t_0) = x_0 \\
\\
\dot{\Phi}(t,t_0) = J_f(x,t)\Phi(t,t_0) \\
\Phi(t_0,t_0) = \mathbb{1}_N
\end{cases}
$$

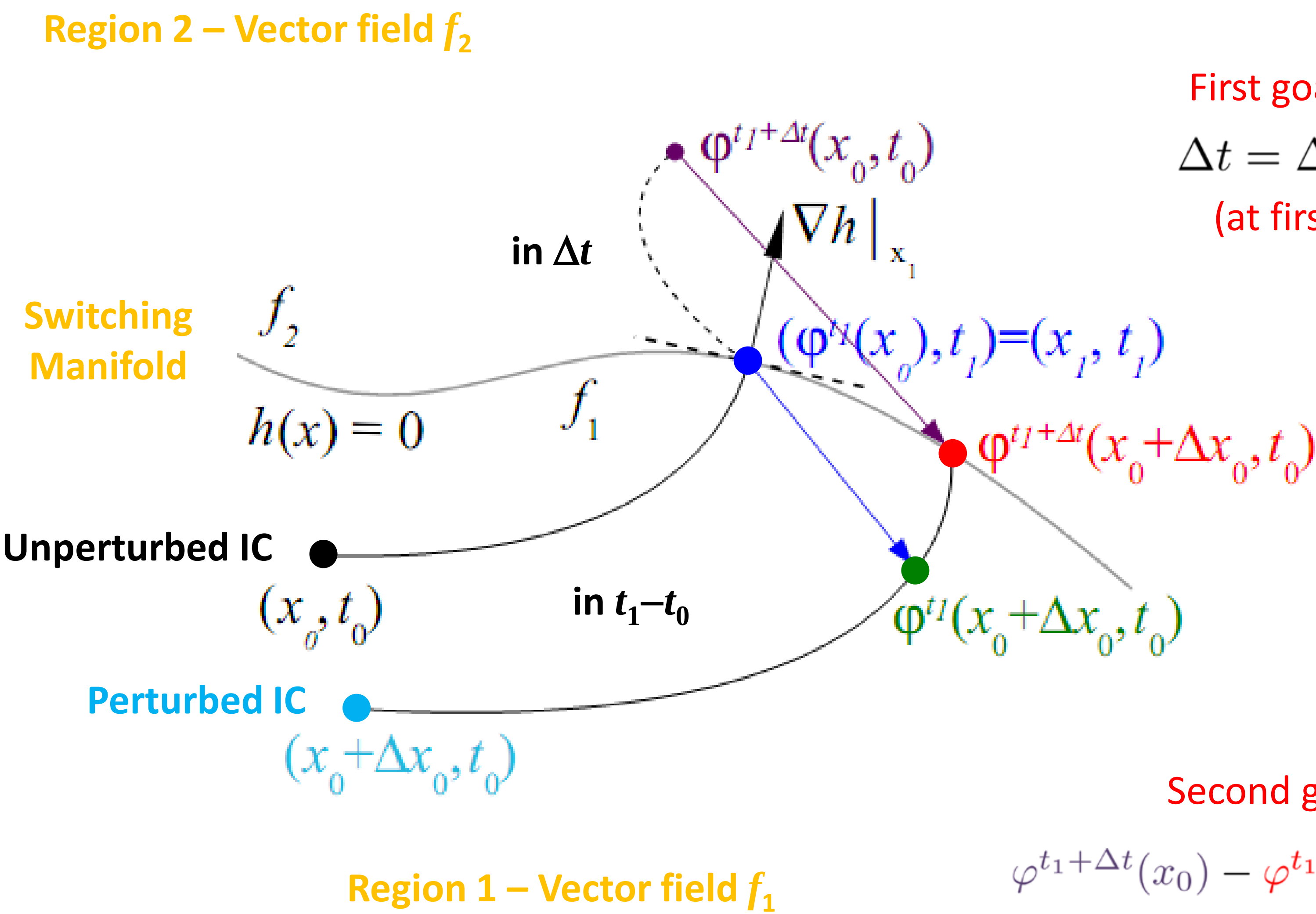
*M. DI BERNARDO, C.J. BUDD, A.R. CHAMPNEYS, AND P. KOWALCZYK, *Piecewise-smooth Dynamical Systems, Theory and Applications*, Springer-Verlag, London, 2008.

Hybrid dynamical systems



$$
\begin{cases}
\dot{x}_1 = -x_2 - (x_1^3 - 0.1x_1) - 0.1z_1 \\
\dot{x}_2 = x_1 \\
\dot{z}_1 = 0 \\
x_1(0) = 0.5 \\
x_2(0) = -0.5 \\
z_1(0) = \mathrm{sign}(x_1(0))
\end{cases}
$$

Saltation Matrix



First goal

$$\Delta t = \Delta t\,(\Delta x_0)$$

(at first order)

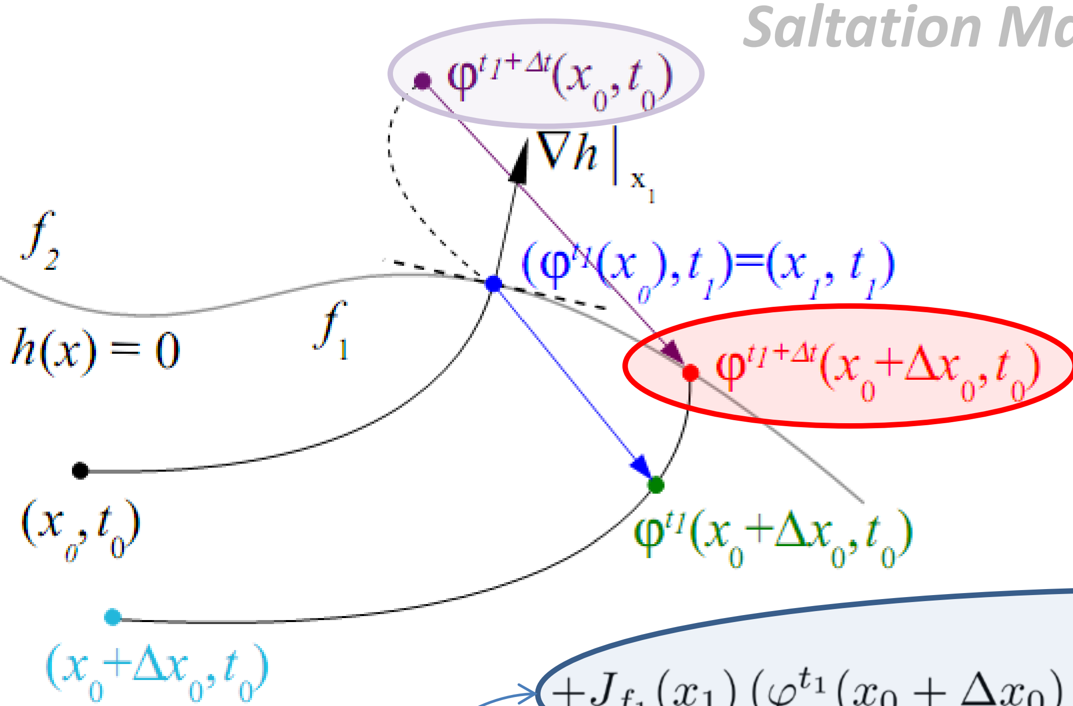
Second goal

$$\varphi^{t_1+\Delta t}(x_0) - \varphi^{t_1+\Delta t}(x_0 + \Delta x_0) =$$

$$= S\,[\varphi^{t_1}(x_0) - \varphi^{t_1}(x_0 + \Delta x_0)]$$

(at first order)

Saltation Matrix



The **purple point** is approximated by an expansion centered in the **blue point**

$$\varphi^{t_1+\Delta t}(x_0) =$$
$$= \varphi^{t_1}(x_0) + f_2\left(\varphi^{t_1}(x_0)\right)\Delta t + O(\Delta t^2) =$$
$$\approx \varphi^{t_1}(x_0) + f_2(x_1)\Delta t$$

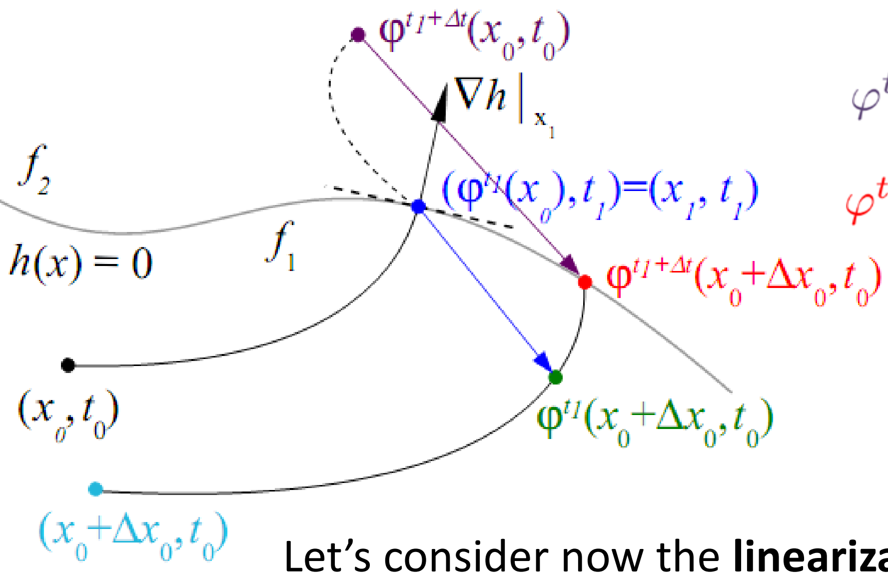
$$f_1(x_1) +$$
$$+ J_{f_1}(x_1)\left(\varphi^{t_1}(x_0 + \Delta x_0) - x_1\right) + O((\varphi^{t_1}(x_0 + \Delta x_0) - x_1)^2)$$

The **red point** is approximated by an expansion centered in the **green point**

$$\varphi^{t_1+\Delta t}(x_0 + \Delta x_0) =$$
$$= \varphi^{t_1}(x_0 + \Delta x_0) + f_1\left(\varphi^{t_1}(x_0 + \Delta x_0)\right)\Delta t + O(\Delta t^2) =$$
$$\approx \varphi^{t_1}(x_0 + \Delta x_0) + f_1(x_1)\Delta t$$

approximated by an
expansion centered in the **blue point**

Saltation Matrix



$$\varphi^{t_1+\Delta t}(x_0) \approx \varphi^{t_1}(x_0) + f_2(x_1)\Delta t$$

$$\varphi^{t_1+\Delta t}(x_0+\Delta x_0) \approx \varphi^{t_1}(x_0+\Delta x_0) + f_1(x_1)\Delta t$$

Let's consider now the **linearization of the manifold** at first in the **red point …**

$$0 = h\left[\varphi^{t_1+\Delta t}(x_0+\Delta x_0)\right] \approx h\left[\varphi^{t_1}(x_0+\Delta x_0) + f_1(x_1)\Delta t\right] \approx$$

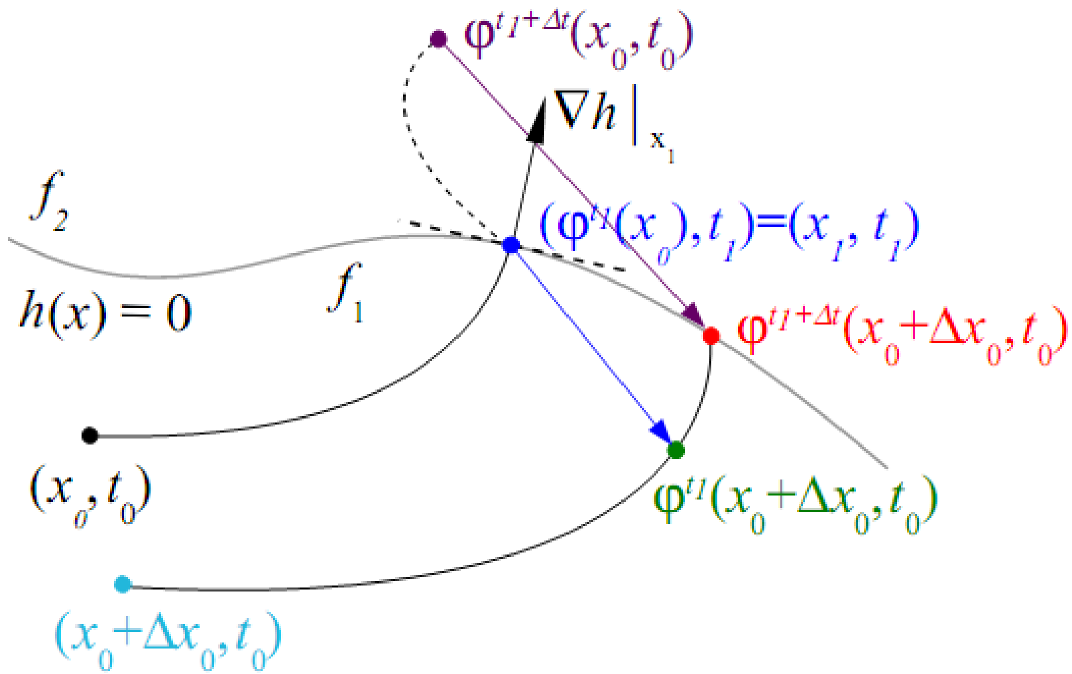
and then in the **green one …**

$$\approx h\left[\varphi^{t_1}(x_0+\Delta x_0)\right] + \nabla^T h\left[\varphi^{t_1}(x_0+\Delta x_0)\right] f_1(x_1)\Delta t \approx$$

$$\begin{aligned}&\approx \nabla^T h\left[\varphi^{t_1}(x_0) + J_{f_1}(\varphi^{t_1}(x_0))\Delta x_0\right] = \\ &= \nabla^T h\left[x_1 + J_{f_1}(x_1)\Delta x_0\right] \approx \\ &\approx \nabla^T h(x_1)\end{aligned}$$

$$\begin{aligned}&\approx h\left[\varphi^{t_1}(x_0)\right] + \\ &\nabla^T h\left[\varphi^{t_1}(x_0)\right]\left[\varphi^{t_1}(x_0+\Delta x_0) - \varphi^{t_1}(x_0)\right] = \\ &= \nabla^T h(x_1)\left[\varphi^{t_1}(x_0+\Delta x_0) - \varphi^{t_1}(x_0)\right]\end{aligned}$$

Saltation Matrix



$$0 \approx \nabla^T h\,(x_1)\left[\varphi^{t_1}(x_0+\Delta x_0) - \varphi^{t_1}(x_0)\right] + \nabla^T h\,(x_1)\, f_1(x_1)\Delta t$$

$$\Delta t\,(\Delta x_0) \approx -\frac{\nabla^T h(x_1)}{\nabla^T h(x_1) f_1(x_1)}\left(\varphi^{t_1}(x_0+\Delta x_0) - \varphi^{t_1}(x_0)\right)$$

Saltation Matrix

$$\varphi^{t_1+\Delta t}(x_0) = \varphi^{t_1}(x_0) + f_2(x_1)\Delta t$$

$$\varphi^{t_1+\Delta t}(x_0+\Delta x_0) = \varphi^{t_1}(x_0+\Delta x_0) + f_1(x_1)\Delta t$$

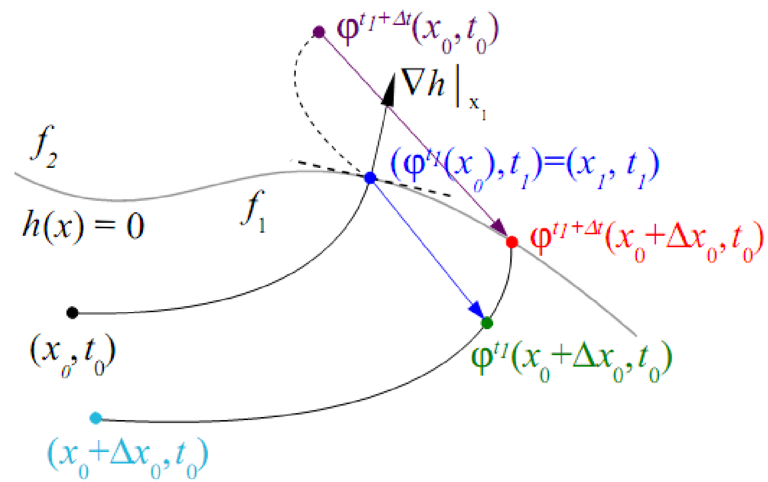
$$\Delta t = -\frac{\nabla^T h(x_1)}{\nabla^T h(x_1) f_1(x_1)}\left(\varphi^{t_1}(x_0+\Delta x_0) - \varphi^{t_1}(x_0)\right)$$

$$\varphi^{t_1+\Delta t}(x_0) - \varphi^{t_1+\Delta t}(x_0+\Delta x_0) = \varphi^{t_1}(x_0) - \varphi^{t_1}(x_0+\Delta x_0) + \left(f_2(x_1) - f_1(x_1)\right)\Delta t$$

$$\varphi^{t_1+\Delta t}(x_0) - \varphi^{t_1+\Delta t}(x_0+\Delta x_0) =$$

$$= \boxed{\left[\mathbb{1} + \frac{\left(f_2(x_1) - f_1(x_1)\right)\nabla^T h(x_1)}{\nabla^T h(x_1) f_1(x_1)}\right]}\left(\varphi^{t_1}(x_0) - \varphi^{t_1}(x_0+\Delta x_0)\right)$$

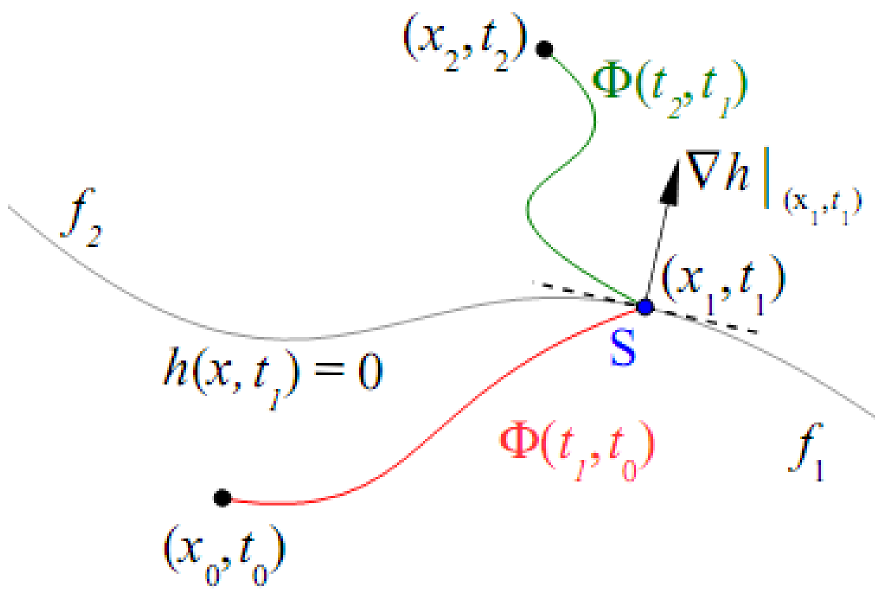
Saltation matrix S



Saltation Matrix

switching vector field and time-varying manifold

$$S = \mathbb{1} + \frac{f_2(x_1,t_1) - f_1(x_1,t_1)}{\nabla^T h(x_1) f_1(x_1,t_1) + \left.\frac{\partial h}{\partial t}\right|_{(x_1,t_1)}} \nabla^T h(x_1,t_1)$$



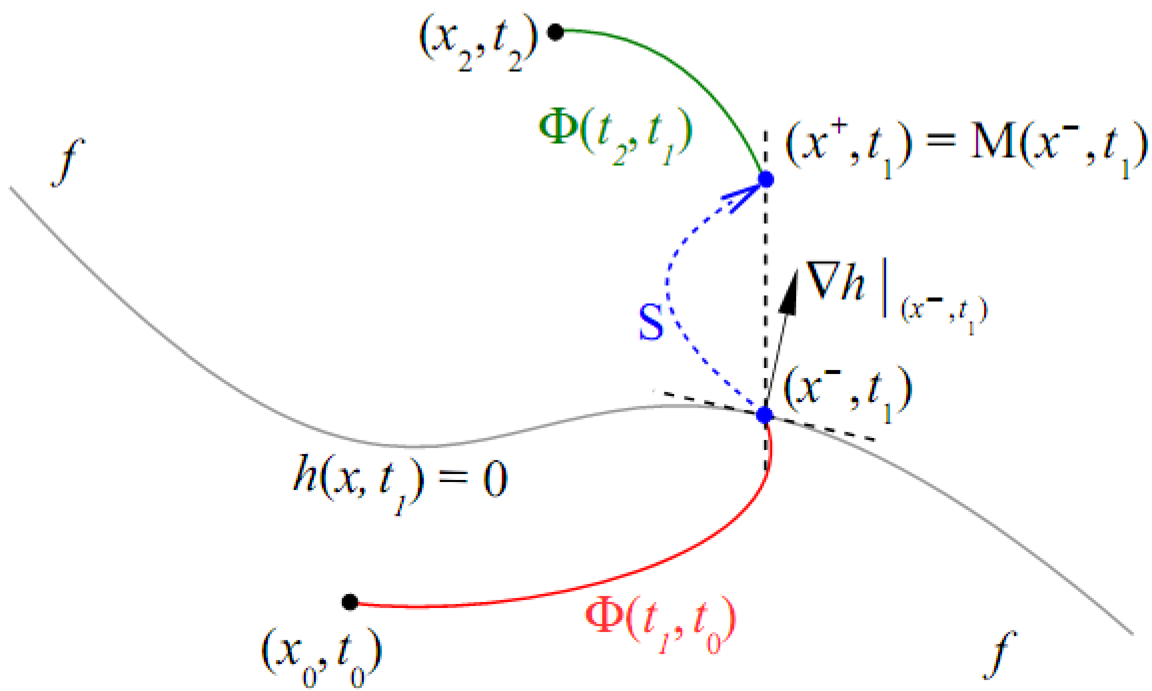
$$\Phi(t_2,t_0) = \Phi(t_2,t_1)\, S\, \Phi(t_1,t_0)$$

Saltation Matrix

impact and time-varying manifold

Jacobian of the mapping function

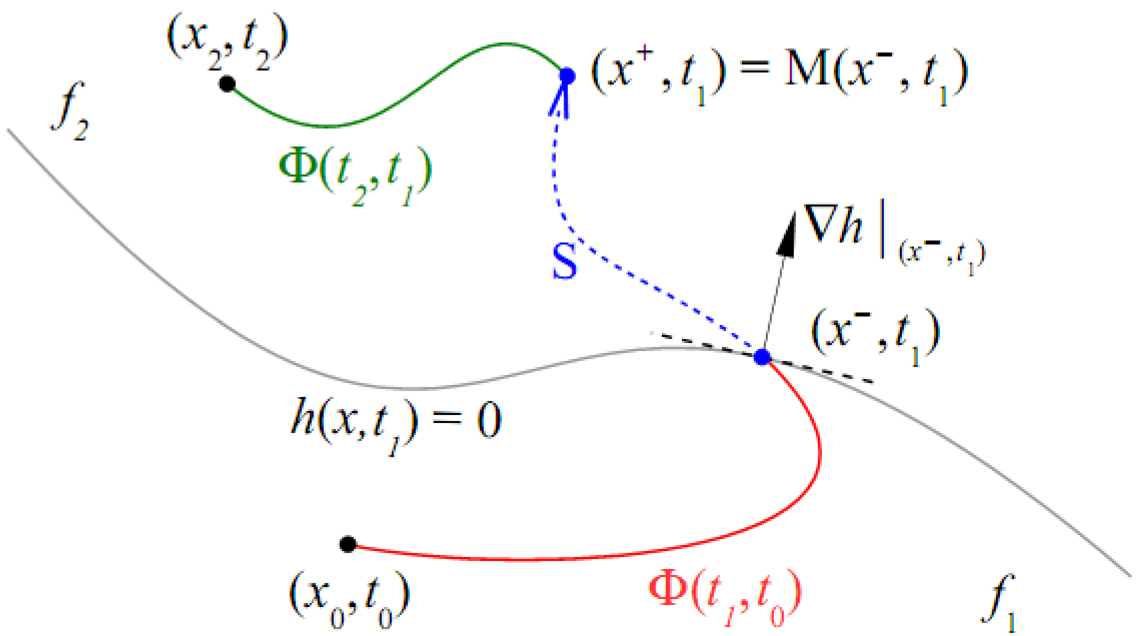
$$S = J_M(x^-, t_1) + \frac{f_1(M(x^-, t_1)) - J_M(x^-, t_1) f_1(x^-, t_1)}{\nabla^T h(x^-, t_1) f_1(x^-, t_1) + \left.\frac{\partial h}{\partial t}\right|_{x^-, t_1}} \nabla^T h(x^-, t_1)$$



Saltation Matrix

***The most general case** (switching, impact and time-varying)*

$$S = J_M(x^-, t_1) + \frac{f_2(M(x^-, t_1)) - J_M(x^-, t_1) f_1(x^-, t_1)}{\nabla^T h(x^-, t_1) f_1(x^-, t_1) + \frac{\partial h}{\partial t}\Big|_{x^-, t_1}} \nabla^T h(x^-, t_1)$$



In our example the Saltation matrix is given by …

$$\begin{cases} \dot{x}_1 = -x_2 - (x_1^3 - 0.1x_1) - 0.1z_1 \\ \dot{x}_2 = x_1 \\ \dot{z}_1 = 0 \\ x_1(0) = 0.5 \\ x_2(0) = -0.5 \\ z_1(0) = \mathrm{sign}(x_1(0)) \end{cases}$$

$$h(x_1, x_2, z_1) = x_1 = 0 \qquad \nabla h = [1, 0, 0]^T$$

$$\begin{bmatrix} x_1^+ \\ x_2^+ \\ z_1^+ \end{bmatrix} = M(x_1^-, x_2^-, z_1^-) = \begin{bmatrix} x_1^- \\ x_2^- \\ -z_1^- \end{bmatrix}$$

The variable z_1 plays the role of a digital state variable

$$J_M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$S = J_M(x^-, t_1) + \frac{f_1(M(x^-, t_1)) - J_M(x^-, t_1) f_1(x^-, t_1)}{\nabla^T h(x^-, t_1) f_1(x^-, t_1) + \left.\frac{\partial h}{\partial t}\right|_{x^-, t_1}} \nabla^T h(x^-, t_1)$$

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \frac{\begin{bmatrix} x_2^- - 0.1(-z_1^-) \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_2^- - 0.1z_1^- \\ 0 \\ 0 \end{bmatrix}}{[1, 0, 0] \begin{bmatrix} x_2^- - 0.1z_1^- \\ 0 \\ 0 \end{bmatrix}} [1, 0, 0]$$

Or depending on the model we choose by …

$$\begin{cases} \dot{x}_1 = -x_2 - (x_1^3 - 0.1x_1) - 0.1\mathrm{sign}(x_1) \\ \dot{x}_2 = x_1 \\ x_1(0) = 0.5 \\ x_2(0) = -0.5 \end{cases}$$

$$h(x_1, x_2) = x_1 = 0$$

$$\nabla h = [1, 0]^T$$

$$S = \mathbb{1} + \frac{(f_2(x_1) - f_1(x_1))\,\nabla^T h(x_1)}{\nabla^T h(x_1) f_1(x_1)}$$

$$S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{\begin{bmatrix} x_2^- - 0.1\mathrm{sign}(x_1^+) \\ 0 \end{bmatrix} - \begin{bmatrix} x_2^- - 0.1\mathrm{sign}(x_1^-) \\ 0 \end{bmatrix}}{[1, 0]\begin{bmatrix} x_2^- - 0.1\mathrm{sign}(x_1^-) \\ 0 \end{bmatrix}}[1, 0]$$

ENGINEERING APPLICATIONS OF ELECTRIC SYSTEMS SIMULATION
A primer on switching electrical circuit simulation
(Practice)

Federico Bizzarri

Dipartimento di Elettronica e Informazione, Politecnico di Milano
P.za Leonardo da Vinci 32, I-20133 Milano, Italy
e-mail: bizzarri@elet.polimi.it

Summary

We will study the dynamics of a DC-DC switching converter

International Journal of Bifurcation and Chaos, Vol. 15, No. 5 (2005) 1549–1578

BIFURCATIONS IN *DC–DC* SWITCHING CONVERTERS: REVIEW OF METHODS AND APPLICATIONS

A. EL AROUDI*, M. DEBBAT and R. GIRAL
Departament d'Enginyeria Electrònica, Elèctrica i Automàtica,
Universitat Rovira i Virgili, Campus Sesceledes,
43007, Tarragona, Spain
**abdelali.elaroudi@urv.net*

G. OLIVAR[†], L. BENADERO[‡] and E. TORIBIO[‡]
[†]*Departament de Matemàtica Aplicada IV,*
[‡]*Departament de Física Aplicada,*
Universitat Politècnica de Catalunya,
Barcelona, Spain

1. Introduction

The elementary *DC–DC* converters buck, boost and buck-boost are a family of Power Electronics (PE) circuits that allow the conversion of electrical energy from one level to another without taking into account, theoretically, losses in the components. They are extensively used in power supplies for electronic circuits and in the control of the flow of energy between DC to DC systems, and in any industrial application where there is a need of stabilizing an output voltage to a desired value. They are also widely used in small spacecrafts such as satellites where DC power is generated by solar arrays. Figure 1 shows the three basic power converters buck, boost and buck-boost.

The operation of *DC–DC* converters is mainly based on the switching between different linear configurations. This must be implemented with an appropriate control of the switches. In a noise perturbation free environment, given the desired output voltage, the switching frequency can be selected and the switches can be turned ON and OFF according to a fixed pattern; this is referred to as the open loop system. In contrast, in industrial applications, noise and perturbations are always present, and also the parameters of the circuits may

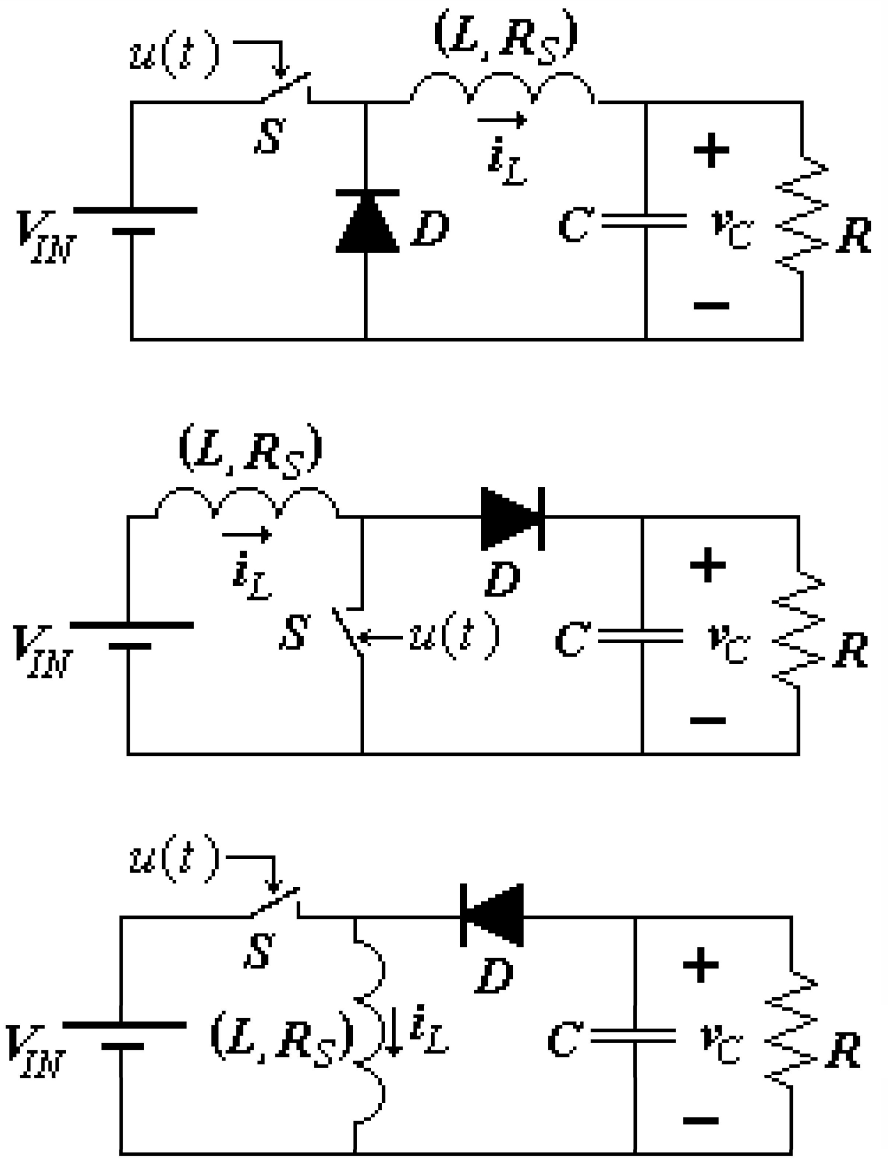


Fig. 1. The three basic power electronic converters buck, boost and buck-boost.

be affected by external disturbances. Thus the use of an appropriate control to counteract the deviations on the output voltage in the system is needed; this is referred to as the closed loop system. We will refer to the ON phase when the switch S is closed and diode D is open; the OFF phase refers to when the switch S is open and diode D closed. A third phase (OFF') that takes place when both switches are open is also possible. The operation mode when this topology takes place is called Discontinuous Conduction Mode (DCM). Otherwise, it is called Continuous Conduction Mode (CCM).

The most interesting dynamics of these systems, from a classical design point of view, is the T-periodic orbit (periodic evolution with the same period as the modulating signal).

Switching *DC–DC* converters are variable structure systems (VSS) that are highly nonlinear. During each phase the electrical switches select the corresponding configuration of the circuit making the energy flow from the input to the output in agreement with the driving signal. The form of this driving signal gives operating flexibility to the circuits and allows, for example, regulation in front of its parameter variations, a task that cannot be done with a rectifier implemented with only diodes. However this switching gives rise also to switched waveforms which can result in a great electromagnetic noise emission.

DC–DC switching converters working in voltage-voltage (V–V) mode get energy from a primary continuous voltage source V_{IN}, and transfer it in the form of a voltage $v_C(t)$ which contains a continuous component V_C and a small ripple. To maintain constant the averaged value of the output voltage, the switching action should be periodic. Thus the modulating signal should also be periodic and the permanent regime will be ordinarily periodic. However, due to their switching action and feedback they are able to present a great variety of nonlinear behavior such as bifurcations and chaos.

A canonical cell of first order with a capacitor in parallel with the load to filter the output ripple is a second order circuit, and its state variables are the inductor current i_L and the capacitor voltage v_C. The signal driving the electrical switch presents different intervals per cycle: for each interval, the circuit takes a specific configuration delivering energy during some intervals and absorbing it within others. As it can be shown, the dynamics of *DC–DC* converter circuits working in continuous conduction mode (CCM) may be described by two independent and first order differential equations for each switch position. The control forces the system to switch between two basic linear configurations. These systems are therefore piecewise linear (PWL). During the first phase, the state space evolution is given by a system of linear equations, i.e.:

$$\dot{x} = A_1 x + B_1 \tag{1}$$

where A_1 and B_1 are constant matrices and x is the vector of the state variables composed by the voltage and current of the energy storage elements like capacitors and inductors. This model follows the evolution of the system for some time and then switches to another linear set of equations in the following form

$$\dot{x} = A_2 x + B_2 \tag{2}$$

with, generally, new constant A_2 and B_2 matrices. In compact form, the dynamical behavior of the system can be described by

$$\dot{x} = f(x,t) = \begin{cases} A_1 x + B_1 & \text{if } \sigma(x,t) > 0 \\ A_2 x + B_2 & \text{if } \sigma(x,t) < 0 \end{cases} \tag{3}$$

As the A's and the B's matrices and vectors are generally different we have a PWL discontinuous vector field f. Other systems can be modeled by similar PWL but continuous vector field f [Freire *et al.*, 1998; Freire *et al.*, 2002]. In general, these discontinuous changes in the constant matrices are responsible for inducing nonlinear effects. However, as the system is PWL, we can solve the time evolutions exactly. In other words, we can find the mapping function that takes the state space variables just after one switching instant up to their values just before the next one. Let us assume that when the switching occurs the value of the variable x takes on certain value $x(t_S)$ at the switching instant t_S. The time derivatives then show discontinuous changes at the switching, i.e.:

$$\dot{x}(t_S^+) - \dot{x}(t_S^-) = (A_2 - A_1)x(t_S) + (B_2 - B_1) \tag{4}$$

where t_S^+ and t_S^- are the time instants just before and just after the switching instant t_S. When the switching condition is time dependent, we have a *nonautonomous* system, which as we know, requires at least one more state space dimension. For feedback systems, as for most of *DC–DC* switching converters, the switching instant depends on the history of the state variables themselves. Therefore, we effectively have a nonlinear system. Hence this

Table 1. The A's and B's matrix for the basic converters during phases ON, OFF and OFF$'$.

Converter	A_{ON}	A_{OFF}	$A_{\mathrm{OFF}'}$	B_{ON}	B_{OFF}	$B_{\mathrm{OFF}'}$
buck	A_a	A_a	A_c	B_a	B_b	B_b
boost	A_b	A_a	A_c	B_a	B_a	B_b
buck-boost	A_b	A_a	A_c	B_a	B_b	B_b

kind of piecewise model, at least in principle, may present nonlinear phenomena such as bifurcations and chaos.

The differential equations, modeling each one of the three configurations that use each converter, can be derived by using the standard Kirchoff's laws. Let us define the matrices A_a, A_b, A_c, B_a and B_c as follows:

$$A_a = \begin{pmatrix} -\dfrac{1}{RC} & \dfrac{1}{C} \\ -\dfrac{1}{L} & -\dfrac{R_S}{L} \end{pmatrix}, \quad A_b = \begin{pmatrix} -\dfrac{1}{RC} & 0 \\ 0 & -\dfrac{R_S}{L} \end{pmatrix},$$

$$A_c = \begin{pmatrix} -\dfrac{1}{RC} & 0 \\ 0 & 0 \end{pmatrix}, \quad B_a = \begin{pmatrix} 0 \\ \dfrac{V_{\mathrm{IN}}}{L} \end{pmatrix},$$

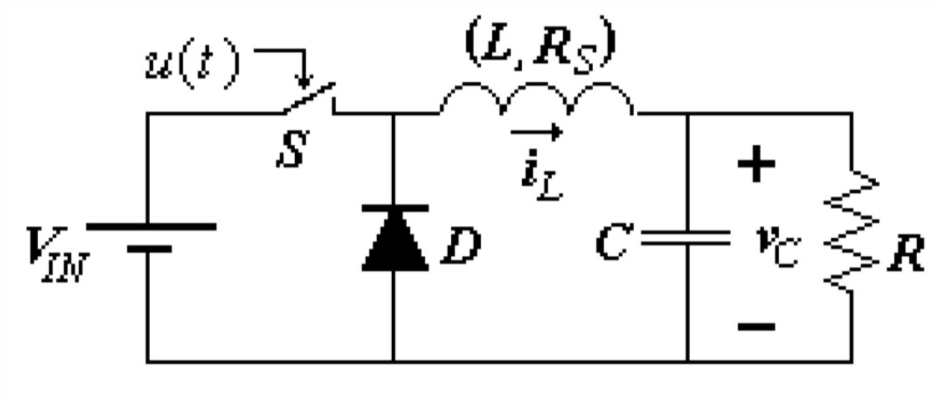
$$B_b = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{5}$$

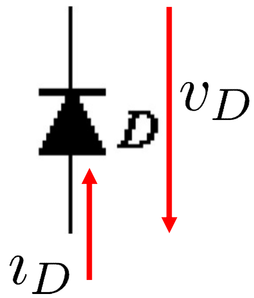
where R is the output load resistance, L is the inductance which is supposed to have an Equivalent Series Resistance ESR R_S, C is the capacitance, and V_{IN} is the input voltage. During each phase (ON, OFF and OFF$'$), and until a switching condition is fulfilled, the dynamics of the system is described as:

$$\dot{x} = Ax + B \tag{6}$$

$x = (v_C, i_L)^T$ is the vector of the state variables and the overdot stands for derivation with respect to time t ($\dot{x} = dx/dt$). Table 1 shows the A's and B's matrices for the three basic converters buck, boost and buck-boost during each phase.

State equations of the buck converter





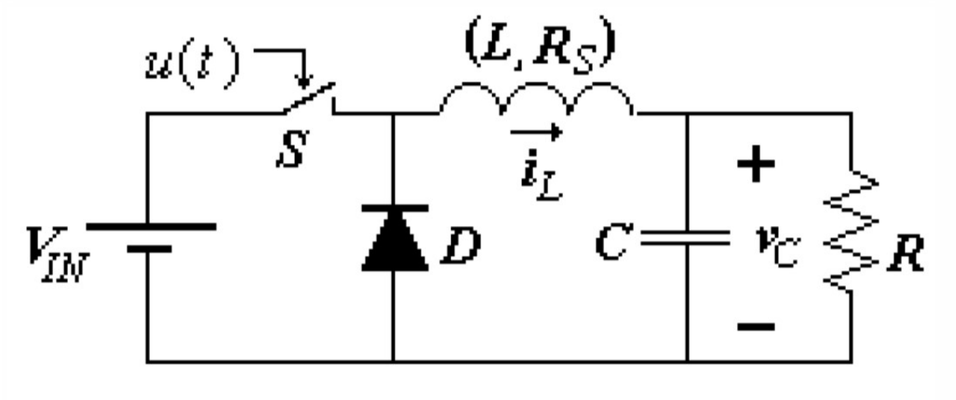
$$\begin{cases} \imath_D = 0, v_D < 0 \\ v_D = 0, \imath_D > 0 \end{cases}$$

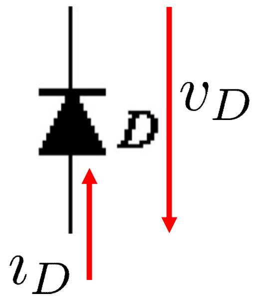
this is referred to as the closed loop system. We will refer to the ON phase when the switch S is closed and diode D is open; the OFF phase refers to when the switch S is open and diode D closed. A third phase (OFF') that takes place when both switches are open is also possible. The operation mode when this topology takes place is called Discontinuous Conduction Mode (DCM). Otherwise, it is called Continuous Conduction Mode (CCM).

$$\begin{cases} V_{IN} - L\dot{\imath}_L - \imath_L R_S - v_C = 0 \\ C\dot{v}_C - \imath_L + \frac{v_C}{R} = 0 \end{cases}$$

$$\begin{bmatrix} \dot{v}_C \\ \dot{\imath}_L \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R_S}{L} \end{bmatrix} \begin{bmatrix} v_C \\ \imath_L \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{V_{IN}}{L} \end{bmatrix}$$

State equations of the buck converter





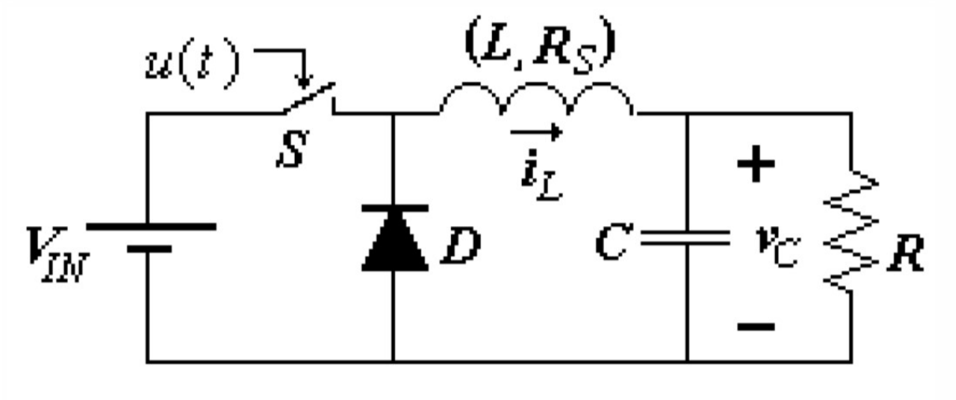
$$\begin{cases} i_D = 0, v_D < 0 \\ v_D = 0, i_D > 0 \end{cases}$$

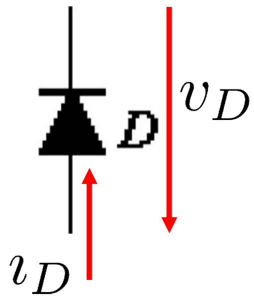
this is referred to as the closed loop system. We will refer to the ON phase when the switch S is closed and diode D is open; the OFF phase refers to when the switch S is open and diode D closed. A third phase (OFF') that takes place when both switches are open is also possible. The operation mode when this topology takes place is called Discontinuous Conduction Mode (DCM). Otherwise, it is called Continuous Conduction Mode (CCM).

$$\begin{cases} -L\dot{i}_L - i_L R_S - v_C = 0 \\ C\dot{v}_C - i_L + \frac{v_C}{R} = 0 \end{cases}$$

$$\begin{bmatrix} \dot{v}_C \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R_S}{L} \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

State equations of the buck converter





$$\begin{cases} \imath_D = 0, v_D < 0 \\ v_D = 0, \imath_D > 0 \end{cases}$$

this is referred to as the closed loop system. We will refer to the ON phase when the switch S is closed and diode D is open; the OFF phase refers to when the switch S is open and diode D closed. A third phase (OFF') that takes place when both switches are open is also possible. The operation mode when this topology takes place is called Discontinuous Conduction Mode (DCM). Otherwise, it is called Continuous Conduction Mode (CCM).

$$\begin{cases} \dot{\imath}_L = 0 \\ \imath_L = 0 \\ C\dot{v}_C + \frac{v_C}{R} = 0 \end{cases}$$

$$\begin{bmatrix} \dot{v}_C \\ \dot{\imath}_L \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_C \\ \imath_L \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

A case study

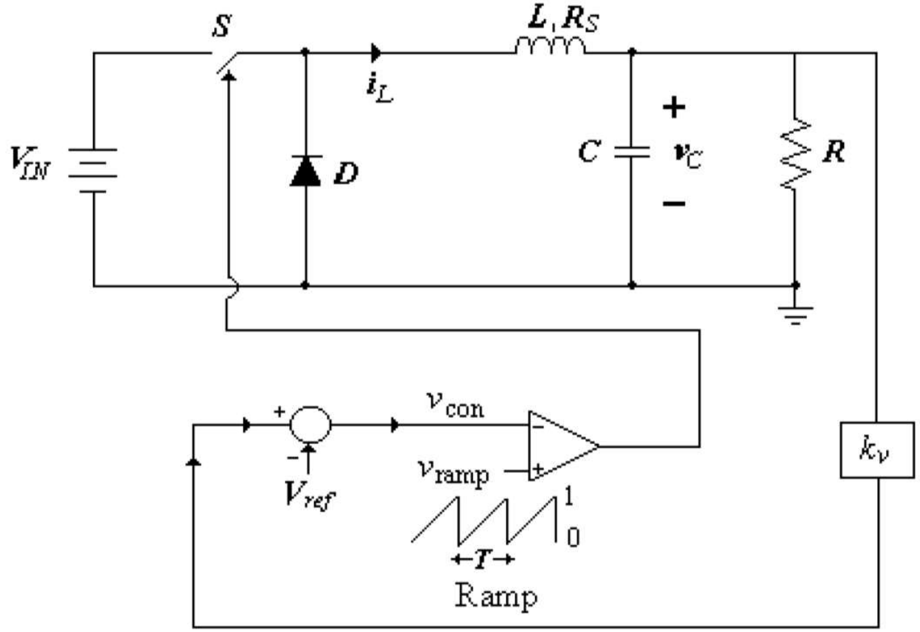


Fig. 8. Schematic diagram of a voltage controlled buck PWM *DC–DC* converter.

6.2.1. *Example 1: PWM voltage controlled buck converter*

Let us consider the PWM controlled buck converter represented in Fig. 8 with the same value of parameters in [Hamill *et al.*, 1992]: $R = 22\,\Omega$, $L = 20\,\text{mH}$, $C = 47\,\mu\text{F}$, $R_S = 0\,\Omega$, $T = 400\,\mu\text{s}$. If a ramp signal with lower value $V_l = 0\,\text{V}$ and upper value $V_u = 1\,\text{V}$ is used, the corresponding control parameters are: $k_v = 1.9091$, $k_i = 0\,\Omega$, $V_{\text{ref}} = 22.4364\,\text{V}$. The matri-

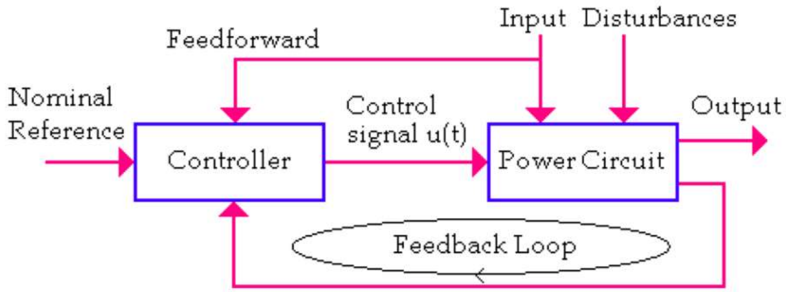


Fig. 3. Block diagram of a *DC–DC* converter with feedback and feedforward.

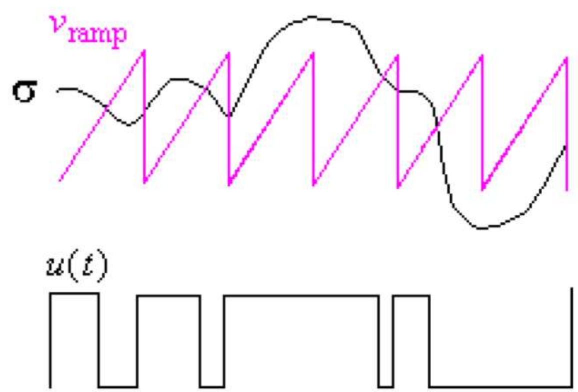


Fig. 4. Control signal $\sigma(t)$, sawtooth ramp signal $v_{\text{ramp}}(t)$ and driving signal $u(t)$.