Exact Relaxation & Global Optimality in ACOPF

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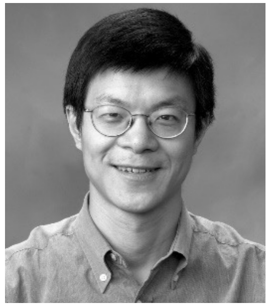

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Summary

OPF is nonconvex & NP hard

OPF is “easy” to solve in practice

- Semidefinite relaxations often exact
- Local algorithms often globally optimal

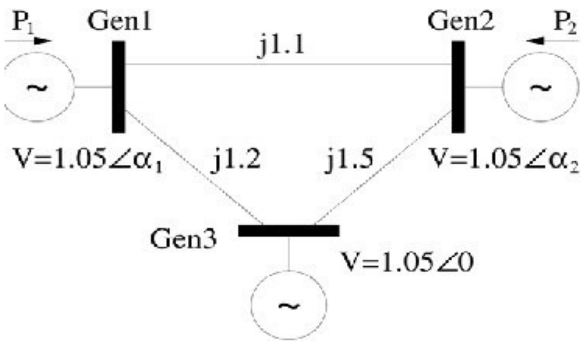
Theoretical support

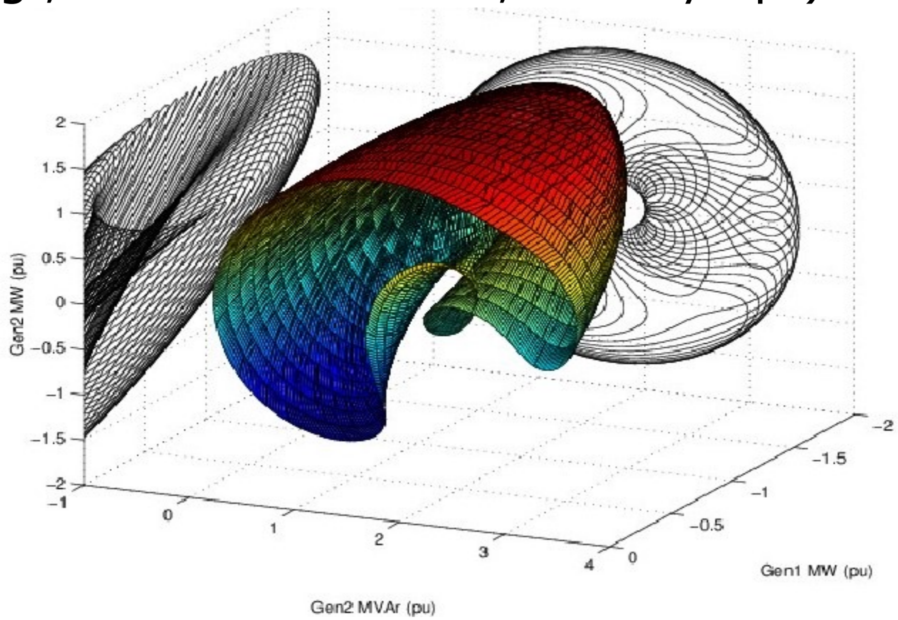
- Exact relaxation
- No spurious local optima

Optimal power flow

OPF is nonconvex

- Nonlinear power flow equations
- Operational constraints, e.g., $V_{min} \leq |V| \leq V_{max}$
- (Discrete variables, e.g., unit commitment, battery opt)





Ian Hiskens, Michigan

Optimal power flow

OPF is NP-hard

- Verma 2009, Bienstock & Verma 2019
- Lavaei & Low 2012
- Lehmann, Grastien & Van Hentenryck 2016

IEEE TRANSACTIONS ON POWER SYSTEMS, VOL. 31, NO. 1, JANUARY 2016

AC-Feasibility on Tree Networks is NP-Hard

Karsten Lehmann, Alban Grastien, and Pascal Van Hentenryck

Reduce NP-hard
subset sum problem to:

Find $\left(\Theta_i, p_{[ij]}, q_{[ij]}\right)$ s.t. power flow equations & constraints

$$\forall i \in N_L : \sum_{[ij]\in E^d} p_{[ij]} = P_i$$

$$\sum_{[ij]\in E^d} q_{[ij]} = Q_i$$

$$\forall i \in N_G : \sum_{[ij]\in E^d} p_{[ij]} \geq 0$$

$$\forall [ij]_b^g \in E^d : p_{[ij]} = g\,(1-\cos(\Theta_i-\Theta_j)) - b\sin(\Theta_i-\Theta_j)$$

$$q_{[ij]} = -b\,(1-\cos(\Theta_i-\Theta_j)) - g\sin(\Theta_i-\Theta_j)$$

$$|\Theta_i-\Theta_j| \leq \overline{\Delta}.$$

Empirical experiences

OPF is "easy" to solve in practice

- Relaxations often exact
- Local solutions often globally optimal

ARPAe NODES quarterly review (Caltech 2013 Sept)

Simulation results

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	#vars	#constrs	IPM (sec)	$\lvert S\rvert^2/vl$	Eig-ratio
IEEE 13-bus	97	40	0.28	1.0000	2.10e-16
IEEE 34-bus	287	120	0.50	1.0000	3.09e-16
IEEE 37-bus	306	126	0.30	1.0000	2.45e-16
IEEE 123-bus	1,030	436	0.41	1.0000	3.31e-16
SCE 47-bus	387	168	0.56	1.0000	2.68e-14
SCE 56-bus	398	173	0.59	1.0000	7.85e-17
SCE Rossi 2145-bus	**16,593**	**6,683**	**2.20**	**0.9997**	**3.71e-16**

SOCP is fast SOCP is exact

Comparison (mesh)

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Test case	Objective values ($/hr)		Running times (sec)		
	SDP/ch	SOCP	SDP	chordal	SOCP
9 bus	5297.4	5297.4	0.2	0.2	0.2
14 bus	8081.7	8075.3	0.2	0.2	0.2
30 bus	574.5	573.6	0.4	0.3	0.3
39 bus	41889.1	41881.5	0.7	0.3	0.3
57 bus	41738.3	41712.0	1.3	0.5	0.3
118 bus	129668.6	129372.4	6.9	0.7	0.6
300 bus	720031.0	719006.5	109.4	2.9	1.8
2383 bus	1840270	1789500.0	-	1005.6	155.3

SOCP inexact SDP not scalable

SOCP is exact (radial) **SDP is exact (mesh)**

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Empirical experiences

OPF is "easy" to solve in practice

- Relaxations often exact
- Local solutions often globally optimal

ARPAe NODES quarterly review (Caltech 2013 August)

Network	% inc from SDP	% inc from chordal SDP	% inc from SOCP
9-bus (line 3 = 34)	0.32	0.32	7
9-bus (line 3 = 35)	0.20	0.20	6.4
30-bus (line 33 = 7.5) flow_move_factor = 2	0.45	0.45	3.24
30-bus (line 33 = 8)	0.11	0.11	2.29
39-bus (line 2 = 220) flow move_factor = 4	0.01	0.01	0.31

< 0.5%

SQP corrects for the inaccuracies of SOCP

SQP starting from relaxation optimal (mesh)

Local algorithms attain global optimal or is close (<0.5% optimality gap)

Empirical experiences

ELSEVIER

Electric Power Systems Research
Volume 189, December 2020, 106688

Proving global optimality of ACOPF solutions

S. Gopinath [a], H.L. Hijazi [a], T. Weisser [a], H. Nagarajan [a], M. Yetkin [b], K. Sundar [a], R.W. Bent [a]

[a] Los Alamos National Laboratory, Los Alamos, NM, USA
[b] Department of Industrial and Systems Engineering, Lehigh University, Bethlehem, PA, USA

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Empirical experiences

Gopinath et al 2020

Case	SDP-RT root gap	SDP-RT final gap	SDP-RT time	SDP-RT iter	Lasserre relaxation gap1	Lasserre relaxation time1	Lasserre relaxation gap2	Lasserre relaxation time2
pglib_opf_case3_lmbd	0.00	0.00	0.08	0	0.38	0.01	0.00	0.23
pglib_opf_case5_pjm	**0.09**	**0.09**	**0.13**	**0**	**5.22**	**0.01**	**0.00**	**20.87**
pglib_opf_case14_ieee	0.00	0.00	0.50	0	0.00	0.22	0.00	141.99
pglib_opf_case24_ieee_rts	0.00	0.00	0.86	0	0.00	0.19	-	-
pglib_opf_case30_as	0.00	0.00	0.85	0	0.00	0.14	0.00	5924.40
pglib_opf_case30_fsr	0.00	0.00	0.71	0	0.01	0.23	-	-
pglib_opf_case30_ieee	0.00	0.00	0.68	0	0.02	0.17	-	-
pglib_opf_case39_epri	0.00	0.00	1.47	0	0.01	0.23	0.00	1336.31
pglib_opf_case57_ieee	0.00	0.00	1.62	0	0.01	0.40	-	-
pglib_opf_case73_ieee_rts	0.00	0.00	3.54	0	0.00	0.60	-	-
pglib_opf_case89_pegase	0.29	0.29	12.96	0	0.34	1.97	-	-
pglib_opf_case118_ieee	0.03	0.03	3.18	0	0.07	1.29	-	-
pglib_opf_case162_ieee_dtc	**1.57**	**0.45**	**5723.00**	**1**	**1.78**	**5.83**	**-**	**-**
pglib_opf_case179_goc	0.07	0.07	6.98	0	0.07	1.64	-	-
pglib_opf_case200_tamu	0.00	0.00	7.11	0	0.00	1.47	-	-
pglib_opf_case300_ieee	**0.10**	**0.10**	**18.99**	**0**	**1.56**	**3.45**	**-**	**-**
pglib_opf_case3_lmbd__api	**0.93**	**0.93**	**0.10**	**0**	**4.99**	**0.01**	**0.00**	**0.20**
pglib_opf_case5_pjm__api	0.01	0.01	0.20	0	0.30*	0.04*	0.00	19.33
pglib_opf_case14_ieee__api	0.01	0.01	0.62	0	0.02	0.06	0.00	143.71
pglib_opf_case24_ieee_rts__api	**1.03**	**0.03**	**21.42**	**1**	**2.07**	**0.21**	**-**	**-**
pglib_opf_case30_as__api	0.72	0.72	0.82	0	16.19*	0.21*	0.39	9807.31
pglib_opf_case30_fsr__api	0.27	0.27	2.24	0	0.52	0.20	-	-
pglib_opf_case30_ieee__api	0.02	0.02	0.74	0	0.34*	0.17*	-	-
pglib_opf_case39_epri__api	0.16	0.16	0.71	0	0.46	0.26	0.01	1973.40
pglib_opf_case57_ieee__api	0.00	0.00	3.34	0	0.02	0.51	-	-
pglib_opf_case73_ieee_rts__api	**2.13**	**0.75**	**261.14**	**1**	**2.92**	**0.65**	**-**	**-**
pglib_opf_case89_pegase__api	**11.70**	**0.93**	**6013.07**	**3**	**12.12**	**2.27**	**-**	**-**
pglib_opf_case118_ieee__api	**8.44**	**0.99**	**2030.47**	**5**	**11.20**	**1.26**	**-**	**-**
pglib_opf_case162_ieee_dtc__api	**1.26**	**0.26**	**16277.76**	**1**	**1.44**	**5.33**	**-**	**-**
pglib_opf_case179_goc__api	0.54	0.54	9.01	0	0.55	1.35	-	-
pglib_opf_case200_tamu__api	0.00	0.00	39.17	0	0.00	2.05	-	-
pglib_opf_case300_ieee__api	0.07	0.07	21.87	0	0.21	3.45	-	-
pglib_opf_case3_lmbd__sad	0.10	0.10	0.08	0	0.62	0.01	0.00	0.21
pglib_opf_case5_pjm__sad	0.00	0.00	0.20	0	0.00	0.03	0.00	18.36
pglib_opf_case14_ieee__sad	0.11	0.11	0.36	0	0.09	0.10	0.00	148.34
pglib_opf_case24_ieee_rts__sad	**3.54**	**0.11**	**21.69**	**1**	**2.52**	**0.17**	**-**	**-**
pglib_opf_case30_as__sad	0.21	0.21	0.95	0	0.16	0.25	0.00	6529.10
pglib_opf_case30_fsr__sad	0.02	0.02	0.69	0	0.02	0.19	-	-
pglib_opf_case30_ieee__sad	0.00	0.00	0.84	0	0.00	0.16	-	-
pglib_opf_case39_epri__sad	0.02	0.02	1.36	0	0.02	0.24	-	-
pglib_opf_case57_ieee__sad	0.04	0.04	4.14	0	0.04	0.63	-	-
pglib_opf_case73_ieee_rts__sad	**2.13**	**0.33**	**228.97**	**1**	**1.48**	**0.58**	**-**	**-**
pglib_opf_case89_pegase__sad	0.29	0.29	12.35	0	0.32	1.95	-	-
pglib_opf_case118_ieee__sad	**2.49**	**0.18**	**471.40**	**1**	**1.83**	**1.21**	**-**	**-**
pglib_opf_case162_ieee_dtc__sad	**1.38**	**0.28**	**3666.39**	**1**	**1.79**	**5.63**	**-**	**-**
pglib_opf_case179_goc__sad	0.94	0.94	15.01	0	0.91	1.59	-	-
pglib_opf_case200_tamu__sad	0.00	0.00	8.50	0	0.00	1.64	-	-
pglib_opf_case300_ieee__sad	**0.12**	**0.12**	**13.98**	**0**	**1.40**	**3.28**	**-**	**-**

Case	SDP-RT root gap	SDP-RT final gap	SDP-RT time	SDP-RT iter	Best of [20] OBBT+cuts gap	Best of [20] time1	Best of [20] final gap	Best of [20] time2
nesta_case3_lmbd	0.00	0.00	0.03	0	0.10	0.95	0.09	0.95
nesta_case4_gs	0.00	0.00	0.03	0	0.00	0.03		
nesta_case5_pjm	**0.11**	**0.11**	**0.05**	**0**	**2.11**	**3.26**	**0.10**	**108.39**
nesta_case6_c	0.00	0.00	0.03	0	-	-	-	-
nesta_case6_ww	0.00	0.00	0.06	0	0.01	1.08		
nesta_case9_wscc	0.00	0.00	0.05	0	0.00	0.09		
nesta_case14_ieee	0.00	0.00	0.09	0	0.00	2.70		
nesta_case24_ieee_rts	0.00	0.00	0.16	0	-	-	-	-
nesta_case29_edin	0.00	0.00	0.73	0	0.01	33.99		
nesta_case30_as	0.00	0.00	0.15	0	0.06	0.11		
nesta_case30_fsr	0.01	0.01	0.17	0	0.07	14.49		
nesta_case30_ieee	0.02	0.02	0.16	0	0.03	14.55		
nesta_case39_epri	0.01	0.01	0.25	0	0.05	0.25		
nesta_case57_ieee	0.00	0.00	0.40	0	0.06	0.22		
nesta_case73_ieee_rts	0.00	0.00	2.32	0	-	-	-	-
nesta_case118_ieee	0.02	0.02	0.97	0	0.14	355.50	0.10	502.47
nesta_case162_ieee_dtc	**0.88**	**0.88**	**8.00**	**0**	**1.57**	**948.30**	**1.46**	**1837.44**
nesta_case189_edin	0.05	0.05	1.08	0	0.04	63.15		
nesta_case300_ieee	0.07	0.07	3.41	0	0.09	520.50		
nesta_case3_lmbd__api	0.32	0.32	0.02	0	0.81	1.05	0.02	3.34
nesta_case4_gs__api	0.02	0.02	0.03	0	0.03	0.55		
nesta_case5_pjm__api	0.00	0.00	0.04	0	0.05	0.81		
nesta_case6_c__api	0.01	0.01	0.04	0	-	-	-	-
nesta_case6_ww__api	0.02	0.02	0.06	0	0.00	3.39		
nesta_case9_wscc__api	0.00	0.00	0.05	0	0.00	0.06		
nesta_case14_ieee__api	0.05	0.05	0.08	0	0.04	13.18		
nesta_case24_ieee_rts__api	0.54	0.54	0.20	0	-	-	-	-
nesta_case29_edin__api	0.00	0.00	2.38	0	0.04	136.83		
nesta_case30_as__api	0.29	0.29	0.17	0	0.09	62.09		
nesta_case30_fsr__api	**4.93**	**2.66**	**71.69**	**6**	**5.15**	**90.56**	**0.83**	**1802.18**
nesta_case30_ieee__api	0.10	0.10	0.19	0	0.06	60.03		
nesta_case39_epri__api	0.00	0.00	0.53	0	0.01	26.33		
nesta_case57_ieee__api	0.09	0.09	0.43	0	0.06	125.44		
nesta_case73_ieee_rts__api	0.35	0.35	1.10	0	-	-	-	-
nesta_case118_ieee__api	**17.50**	**1.91**	**2981.79**	**12**	**7.83**	**911.90**	**7.83**	**1834.74**
nesta_case162_ieee_dtc__api	**0.84**	**0.84**	**8.03**	**0**	**1.03**	**2007.66**	**1.03**	**2007.68**
nesta_case189_edin__api	0.12	0.12	1.10	0	0.91	592.86	0.12	663.19
nesta_case300_ieee__api	0.00	0.00	53.02	0	0.10	1048.07		
nesta_case3_lmbd__sad	0.11	0.11	0.02	0	0.09	1.29	0.03	1.29
nesta_case4_gs__sad	0.05	0.05	0.03	0	0.01	0.66		
nesta_case5_pjm__sad	0.04	0.04	0.04	0	0.07	0.94		
nesta_case6_c__sad	0.01	0.01	0.04	0	-	-	-	-
nesta_case6_ww__sad	0.00	0.00	0.08	0	0.00	1.53		
nesta_case9_wscc__sad	0.03	0.03	0.06	0	0.01	1.14		
nesta_case14_ieee__sad	0.00	0.00	0.09	0	0.06	0.16		
nesta_case24_ieee_rts__sad	**5.13**	**0.34**	**17.95**	**1**	**-**	**-**	**-**	**-**
nesta_case29_edin__sad	**23.21**	**0.81**	**215.20**	**2**	**0.70**	**325.68**	**0.67**	**1837.01**
nesta_case30_as__sad	0.47	0.47	0.25	0	0.09	38.85		
nesta_case30_fsr__sad	0.10	0.10	0.18	0	0.09	26.57		
nesta_case30_ieee__sad	0.03	0.03	0.15	0	0.02	26.78		
nesta_case39_epri__sad	0.05	0.05	0.28	0	0.02	11.54		
nesta_case57_ieee__sad	0.04	0.04	0.42	0	0.07	36.75		
nesta_case73_ieee_rts__sad	**3.42**	**0.73**	**202.15**	**1**	**-**	**-**	**-**	**-**
nesta_case118_ieee__sad	**5.93**	**0.29**	**1062.24**	**2**	**3.35**	**748.42**	**3.07**	**1804.74**
nesta_case162_ieee_dtc__sad	**3.23**	**0.29**	**T.L.**	**2**	**3.76**	**1741.94**		
nesta_case189_edin__sad	**1.23**	**0.25**	**562.08**	**1**	**1.41**	**315.67**	**1.06**	**1814.79**
nesta_case300_ieee__sad	0.09	0.09	3.57	0	0.10	1226.36		

- Semidefinite relaxations are often exact
- Enhancements (e.g., adding valid cuts) closes almost all optimality gaps (<1%)

Summary

OPF is nonconvex & NP hard

OPF is "easy" to solve in practice

- Semidefinite relaxations often exact
- Local algorithms often globally optimal

Theoretical support

- Exact relaxation
- No spurious local optima

Theoretical support

Exact relaxations

- Sufficient condition (radial): angle difference
- Sufficient condition (radial): injection bounds
- Sufficient condition (3phase radial): critical buses

Exact relaxation

Exact relaxation: radial networks

- Caltech [2011, 2012, 2013]: Bose, Chandy, Chen, Gayme, Farivar, Gan, Lavaei, Li, Low, Sojoudi, Topcu
- Berkeley/UIUC [2011, 2013]: Dominguez-Garcia, Lam, Lavaei, Tse, Zhang
- …. Recent surveys: Low 2014, Molzach & Hiskens 2019

Exact relaxation: meshed networks

- Kim & Kojima (Comp Opt App 2003): only real QCQP
- Zhou & Low (CDC 2019): multiphase radial network
- Burer & Ye (Math Prog 2020): diagonal QCQP
- Azuma et al (arXiv 2020): forest-structured network

Global optimality

Theoretical support

- Exact relaxation: known sufficient conditions

- No spurious local optima: widely observed empirically, but no analytical evidence till recently

A Sufficient Condition for Local Optima to be Globally Optimal

Fengyu Zhou and Steven H. Low
Engineering & Applied Science, Caltech, Pasadena CA

CDC 2020

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Setup

$$\begin{array}{lll} \underset{x}{\text{minimize}} & f(x) \\ \text{subject to} & x \in \mathcal{X} \end{array}$$

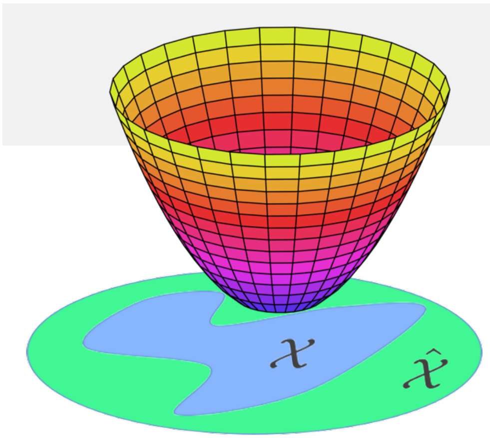
f : continuous, convex

X : compact, nonconvex

Convex relaxation:

$$\begin{array}{lll} \underset{x}{\text{minimize}} & f(x) \\ \text{subject to} & x \in \hat{\mathcal{X}}. \end{array}$$

$\hat{X}$: compact, convex, $X \subseteq \hat{X} \subseteq K^n$



Setup

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & x \in \mathcal{X} \end{aligned}$$

f : continuous, convex

X : compact, nonconvex

Convex relaxation:

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & x \in \hat{\mathcal{X}}. \end{aligned}$$

$\hat{X}$: compact, convex, $X \subseteq \hat{X} \subseteq K^n$

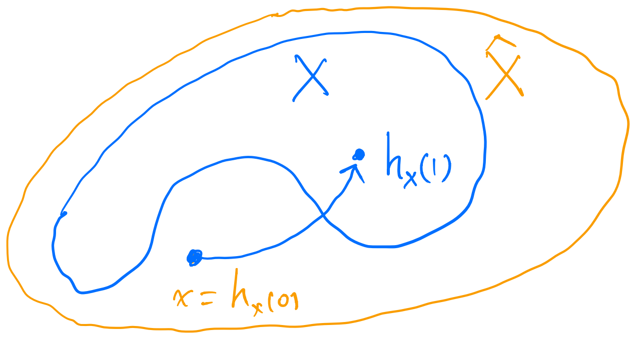
Relaxation (2) is exact if there exists optimal solution of (2) that is optimal for (1)

Key result [Zhou 2020]: Lyapunov-like conditions for

- Relaxation (2) is exact; and
- Any local optimum of (1) is globally optimal

Exact relaxation

Definition: A *path from* $x \in \hat{X} \setminus X$ *to* X is a continuous function $h_x: [0,1] \to \hat{X}$ such that $h_x(0) = x$ and $h_x(1) \in X$



Lemma [Zhou 2020]

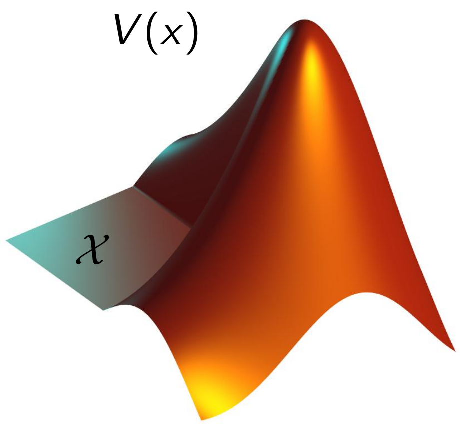
(2) is exact $\Leftrightarrow \forall x \in \hat{X} \setminus X$ there is a path h_x from x to X such that

- $f(h_x(t))$ nonincreasing in t
- $f(h_x(1)) < f(h_x(0))$

Lyapunov-like function

Definition: A *Lyapunov-like function* is a continuous function $V\colon \hat{X} \to \mathbb{R}_+$ such that

$$V(x)\begin{cases} = 0 & x \in X \\ > 0 & x \in \hat{X} \setminus X \end{cases}$$



Lyapunov-like function

Standard Lyapunov function

- Dynamical system: $\dot{y} = f(y(t))$
- Global asymptotic stability: $y(t) \to y^*$
- Stability certificate: Lyapunov function $V(y)$ s.t.
  1. $V(y) > 0$ if $y \neq y^*$, $= 0$ if $y = y^*$
  2. $\dot{V}(y(t)) < 0$ along trajectory $y(t)$

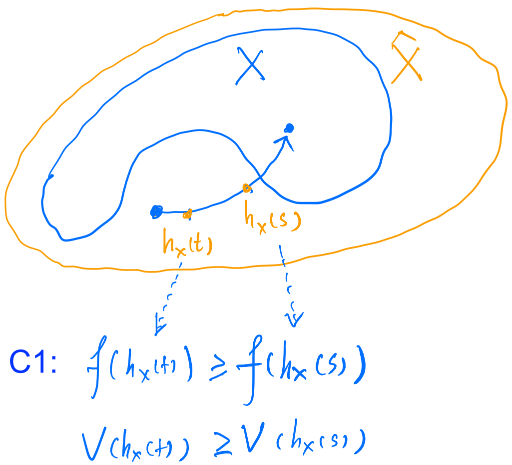
Our case

- Trajectory (path $y(t) = h_x(t)$) is not specified
- Goal is to enter X: $x = y(0) \to y(1) \in X$
- Lyapunov-like $V(y)$ s.t.
  1. $V(y) > 0$ if $y \neq y^*$, $= 0$ if $y = y^*$
  2. $C1$: $V(y(t))$ non-increasing along trajectory $y(t)$
- Cost $f(y(t))$ must be non-increasing along $y(t)$ and $y(1) < y(0) = x$

No spurious local optima

Conditions: $\exists$ paths $\{h_x : x \in \hat{X} \setminus X\}$ and a Lyapunov-like function V such that

- C1: both $f(h_x(t))$ and $V(h_x(t))$ are non-increasing for $t \in [0,1]$, and $f(h_x(0)) > f(h_x(1))$
- C2: $\{h_x : x \in \hat{X} \setminus X\}$ is uniformly bounded and uniformly equicontinuous



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Theorem [Zhou 2020]

- C1, C2 $\Leftarrow$ all local optima of (1) globally optimal & (2) exact

Are C1, C2 sufficient ?

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No spurious local optima

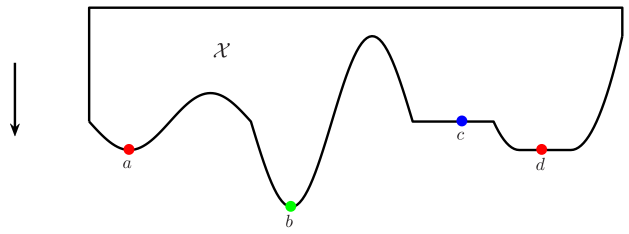
Conditions: ∃ paths $\{h_x : x \in \hat{X} \setminus X\}$ and a Lyapunov-like function V such that

- C1: both $f(h_x(t))$ and $V(h_x(t))$ are non-increasing for $t \in [0,1]$, and $f(h_x(0)) > f(h_x(1))$
- C2: $\{h_x : x \in \hat{X} \setminus X\}$ is uniformly bounded and uniformly equicontinuous
- C3: ∃ $k > 0$ such that $f(h_x(t)) - f(h_x(s)) \geq k\|h_x(t) - h_x(s)\|$

Local algorithm may converge to any local optimum:

Examples

Global optimum (g.o.): *b*
Pseudo local optimum (p.l.o.): *c*
Genuine local optimum (g.l.o.): *a*, *d*



- C1, C2 eliminate genuine local optimal (a, d)
- C3 eliminates pseudo local optimum (c)

No spurious local optima

Conditions: $\exists$ paths $\{h_x : x \in \hat{X} \setminus X\}$ and a Lyapunov-like function V such that

- C1: both $f(h_x(t))$ and $V(h_x(t))$ are non-increasing for $t \in [0,1]$, and $f(h_x(0)) > f(h_x(1))$
- C2: $\{h_x : x \in \hat{X} \setminus X\}$ is uniformly bounded and uniformly equicontinuous
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Theorem [Zhou 2020]

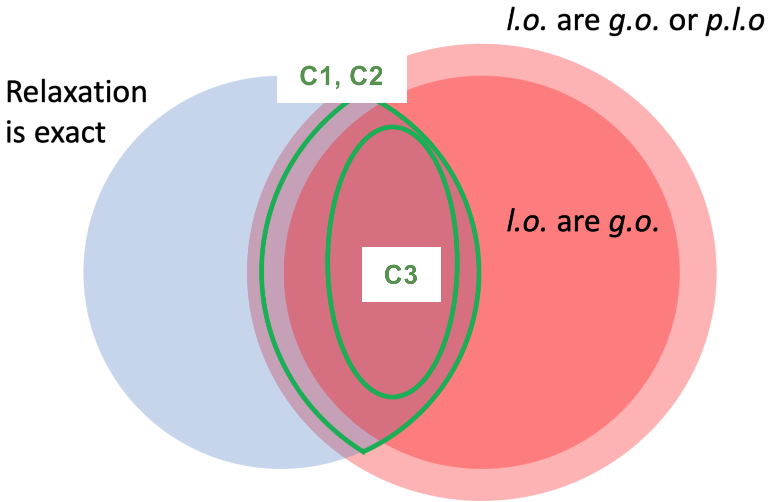
- C1, C2 $\Leftarrow$ all local optima of (1) globally optimal & (2) exact
- C1, C2, C3 $\Rightarrow$ all local optima of (1) globally optimal & (2) exact

Applications: OPF, low rank SDP, …

Global optimality: summary

Conditions: $\exists$ paths $\{h_x : x \in \hat{X} \setminus X\}$ and a Lyapunov-like function V such that

- C1: both $f(h_x(t))$ and $V(h_x(t))$ are non-increasing for $t \in [0,1]$, and $f(h_x(0)) > f(h_x(1))$
- C2: $\{h_x : x \in \hat{X} \setminus X\}$ is uniformly bounded and uniformly equicontinuous
- C3: $\exists\, k > 0$ such that $f(h_x(t)) - f(h_x(s)) \geq k\|h_x(t) - h_x(s)\|$



Application to OPF

Non-convex problem:

$$\begin{aligned} \min_{s,v,\ell,S} \quad & f(s) \\ \text{s.t.} \quad & \text{convex constr.} \\ & v_j \ell_{jk} = |S_{jk}|^2 \end{aligned}$$

Relaxed problem:

$$\begin{aligned} \min_{s,v,\ell,S} \quad & f(s) \\ \text{s.t.} \quad & \text{convex constr.} \\ & v_j \ell_{jk} \geq |S_{jk}|^2 \end{aligned}$$

Construction

$V := \sum_{jk} v_k \ell_{jk} - |S_{jk}|^2$

h_x: linearly decrease ℓ_{jk} and linearly adjust s, S accordingly.

This construction satisfies C1, C2, C3

Theorem

If there are no lower bounds for s_j, i.e., bus injections, then any local optimum of the original non-convex OPF is also a global optimum.

First result on the local optimality for non-convex OPF problem. [Zhou, Low CDC2020]

Baran-Wu 1989 DistFlow model

Application to OPF

Non-convex problem:

$$\begin{aligned}\min_{s,v,\ell,S} \quad & f(s)\\ \text{s.t.} \quad & \text{convex constr.}\\ & v_j \ell_{jk} = |S_{jk}|^2\end{aligned}$$

Relaxed problem:

$$\begin{aligned}\min_{s,v,\ell,S} \quad & f(s)\\ \text{s.t.} \quad & \text{convex constr.}\\ & v_j \ell_{jk} \geq |S_{jk}|^2\end{aligned}$$

Construction (a 2-bus example)

- $V := v_1 \ell_{12} - |S_{12}|^2$
- For $x \in \hat{\mathcal{X}} \setminus \mathcal{X}$, we have $|S_{12}|^2 - v_1\ell_{12} < 0$.
- Let Δ be the positive root of $\frac{|z_{12}|^2}{4} a^2 + \left(v_1 - \mathrm{Re}(z_{12} S_{12}^{\mathrm{H}})\right) a + |S_{12}|^2 - v_1 \ell_{12}$
- Consider the path:

$$\begin{aligned}\tilde{s}_j(t) &= s_j - \frac{t}{2} z_{12}\Delta - \frac{t}{2} z_{12}\Delta,\\ \tilde{v}_j(t) &= v_j,\\ \tilde{\ell}_{12}(t) &= \ell_{12} - t\Delta,\\ \tilde{S}_{12}(t) &= S_{12} - \frac{t}{2} z_{12}\Delta.\end{aligned}$$

Construction satisfies C1, C2, C3

- SOCP relaxation is exact
- Local optima are globally optimal

F. Zhou

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Summary

OPF is nonconvex & NP hard

OPF is “easy” in practice

- Semidefinite relaxations often exact
- Local algorithms often globally optimal

Analytical properties

- Exact relaxation
- No spurious local optima